RATIONAL CATALAN NUMBERS AND CORE PARTITIONS

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Abstract. In this exposition, we explore the rich combinatorial structure underlying (a, b)-core partitions. Starting with the cycle lemma, we build toward deeper results through (a, b)-Dyck paths and the rational Catalan numbers. We then develop James' abacus representation to understand core partitions and prove Anderson's bijection, which connects the size of a core to the area under its corresponding path. Our journey culminates in a derivation, following Armstrong–Hanusa–Jones [2], of the formula for the expected size of an (a, b)-core partition: for coprime positive integers a and b, the average size equals $(a-1)(b-1)(a+b+1)/24$. Along the way, we highlight key ideas from lattice path enumeration, partition theory, and composition sums in an accessible and structured way.

1. Introduction

The study of core partitions reveals intrinsic structure within the theory of integer partitions. Understanding this fundamental structure has profound implications across representation theory and provides a window into the hidden patterns that govern combinatorial objects.

A partition is called an a-core if none of its hook lengths are divisible by a. When we require avoidance of hooks divisible by both a and b where $\gcd(a, b) = 1$, we obtain (a, b)-cores. A fundamental result states that there are only finitely many such partitions, and they are enumerated by the rational Catalan number $\mathrm{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a}$.

The true depth of studying core partitions emerges from their connections with lattice paths. The field was transformed by the discovery of Anderson's bijection [1], which creates a correspondence between (a, b)-cores and (a, b)-Dyck paths. This bijection reveals that the seemingly simple question of computing the average size of a core is intimately connected to computing the average area under a lattice path.

Guided by this perspective, we begin by establishing precise definitions for our study. Section 2 develops the cycle lemma as a counting tool for lattice paths. Section 3 uses this to enumerate (a, b)-Dyck paths via the rational Catalan numbers. Section 4 introduces rim hooks and James' abacus to characterize core partitions. Section 5 proves Anderson's bijection connecting paths to cores. Sections 6 and 7 compute the average size of an (a, b)-core.

Definition 1.1 (Young Diagram)**.** A *Young diagram* is a collection of boxes arranged in left-justified rows, where the i-th row contains λ_i boxes and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$.

Definition 1.2 (Hook Length)**.** The *hook length* of a cell (i, j) in a Young diagram, denoted $h_{i,j}$, is the number of cells directly to the right of (i, j), plus the number directly below, plus 1 (for the cell itself).

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To illustrate these concepts, consider the partition $\lambda = (4, 3, 2)$. We can verify that the cell $(1, 1)$ has hook length 6 by counting: 3 cells to the right, 2 cells below, plus 1. Figure 1 shows all hook lengths for this partition.

2	1		
4	2	1	
6	4	2	1

Figure 1. Hook lengths of the partition $(4, 3, 2)$.

In this paper, we establish the following main result.

Theorem 1.3 (Main Result)**.** *For coprime positive integers a and b,*

$$\mathbb{E}[\textit{size of an } (a,b)\textit{-core}] = \frac{(a-1)(b-1)(a+b+1)}{24} \tag{1.1}$$

where the expectation is taken uniformly over all (a, b)-core partitions.

2. The Cycle Lemma

We first develop a counting tool called the cycle lemma to enumerate (a, b)-Dyck paths. The motivation for this lemma comes from viewing sequences arranged in a circle rather than a line.

Definition 2.1 (Cyclic Rotation)**.** Let $w = w_1 w_2 \cdots w_n$ be a word. The i-th *cyclic rotation* of w is the word $w_i w_{i+1} \cdots w_n w_1 \cdots w_{i-1}$ obtained by starting at position i and reading around the circle.

We will illustrate this by taking the following example. Consider the word $w = EENENE$ with 3 copies of E and 3 copies of N. The 6 cyclic rotations are:

Position 1: $EENENE$
Position 2: $ENENEE$
Position 3: $NENEEE$
Position 4: $ENEEEN$
Position 5: $NEEENE$
Position 6: $EEENEN$

Lemma 2.2 (Cycle Lemma)**.** *Let w be a word with a copies of letter E and b copies of letter N, where $\gcd(a, b) = 1$. Among the $a + b$ cyclic rotations of w, exactly one produces a path that stays weakly below the line from $(0, 0)$ to (a, b).*

Proof. We assign weight $+b$ to each letter E and weight $-a$ to each letter N. If we interpret E as an East step $(1, 0)$ and N as a North step $(0, 1)$, then after reading a prefix that takes us to point (x, y), the cumulative weighted sum equals $bx - ay$.

The path stays below the line $y = \frac{b}{a}x$ precisely when $bx - ay \geq 0$ at every point along the path. Equivalently, all weighted partial sums must be non-negative.

Since we have a letters of weight b and b letters of weight $-a$, the total weight is $ab - ab = 0$. As we traverse the circular arrangement and track cumulative weights, the sum fluctuates but returns to zero after a complete circuit.

For existence, consider where this cumulative sum achieves its global minimum. If we start reading immediately after this minimum position, every subsequent partial sum (relative to our new starting point) is non-negative. This is because we begin at the lowest point and eventually return to zero, so we never dip below our starting value.

For uniqueness, suppose two different positions both achieve the global minimum. Consider the segment of the word between these positions. This segment is nonempty and a proper subset of the word, containing a' letters E and b' letters N with $0 < a' < a$ and $0 < b' < b$. The total weight is $a'b - b'a = 0$. This means $a'/b' = a/b$, and since $\gcd(a, b) = 1$, we must have $a' = ca$ and $b' = cb$ for some positive integer c. But $a' < a$ forces $c < 1$, a contradiction. ∎

3. Rational Catalan Numbers

We now translate the cycle lemma into a formula for counting Dyck paths. This geometric viewpoint will be crucial when we connect to partition theory.

Definition 3.1 ((a, b)-Dyck Path)**.** An (a, b)-*Dyck path* is a lattice path from $(0, 0)$ to (a, b) consisting of a East steps $(1, 0)$ and b North steps $(0, 1)$ that stays weakly below the line $y = \frac{b}{a}x$.

To illustrate, consider the case $(a, b) = (3, 2)$. We need paths from $(0, 0)$ to $(3, 2)$ that stay below $y = \frac{2}{3}x$. Let us check which of the $\binom{5}{3} = 10$ arrangements are valid:

- $EEENN$: Visits $(1, 0) \to (2, 0) \to (3, 0) \to (3, 1) \to (3, 2)$. Valid.
- $EENEN$: Visits $(1, 0) \to (2, 0) \to (2, 1) \to (3, 1) \to (3, 2)$. Valid.
- $ENEEN$: Reaches $(1, 1)$ where $1 > \frac{2}{3}$. Invalid.
- $NEEEN$: Reaches $(0, 1)$ where $1 > 0$. Invalid.

Among the 10 arrangements, we show a few representative cases. After checking all, we find exactly 2 valid paths as shown in Figure 2.

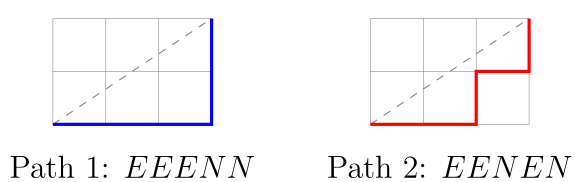


Figure 2. The two $(3, 2)$-Dyck paths.

Theorem 3.2 (Rational Catalan Numbers)**.** *For coprime positive integers a and b, the number of (a, b)-Dyck paths is*

$$\mathrm{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a}. \tag{3.1}$$

Proof. There are $\binom{a+b}{a}$ total ways to arrange a East steps and b North steps. By Lemma 2.2, these arrangements partition into equivalence classes under cyclic rotation. Each equivalence class has exactly $a + b$ elements (the coprimality condition ensures no word has a smaller period), and exactly one element per class gives a valid Dyck path.

Therefore, the number of valid paths equals the number of equivalence classes:

$$\mathrm{Cat}(a,b) = \frac{\binom{a+b}{a}}{a+b} = \frac{1}{a+b}\binom{a+b}{a}.$$

■

The numbers $\mathrm{Cat}(a,b)$ are called the *rational Catalan numbers*. When $b = a+1$, the line $y = \frac{a+1}{a}x$ has slope just above 1, and the $(a, a+1)$-Dyck paths are exactly the classical Dyck paths staying below the diagonal $y = x$. In this case, $\mathrm{Cat}(a, a+1) = C_a = \frac{1}{a+1}\binom{2a}{a}$ reduces to the classical Catalan numbers that count binary trees, balanced parentheses, and over 200 other combinatorial structures catalogued in [3].

4. Rim Hooks and the Abacus

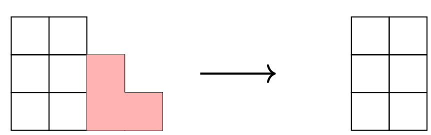
The connection between core partitions and lattice paths comes from James' abacus, a visual representation that makes hook structure transparent. Before introducing the abacus, we must understand rim hooks.

Definition 4.1 (Rim Hook)**.** A *rim hook* of length k in a Young diagram is a connected sequence of k boundary cells that:

(1) Contains no 2×2 square of cells
(2) Can be removed to leave a valid Young diagram

The key property connecting hooks to rim hooks is: a partition has a hook of length k at some cell if and only if a k-rim hook can be removed from the partition.

To illustrate rim hook removal, consider the partition $(4,3,2)$ in Figure 3. The shaded cells form a 3-rim hook. Removing it gives the partition $(2,2,2)$.



$(4,3,2)$ with 3-rim hook

$(2,2,2)$ after removal

Figure 3. Removing a 3-rim hook from $(4,3,2)$ gives $(2,2,2)$.

Definition 4.2 (a-Core)**.** A partition is an *a-core* if it has no hook lengths divisible by a, equivalently, if no a-rim hook can be removed.

With the definition of rim hooks established, we turn to the abacus, which provides an efficient way to represent partitions and understand their core properties.

Definition 4.3 (Beta-Numbers)**.** Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, the *beta-numbers* are $\beta_i = \lambda_i + k - i$ for $i = 1, \ldots, k$. These form a strictly decreasing sequence of non-negative integers.

Example. For $\lambda = (4, 3, 2)$ with $k = 3$ parts:

$$\beta_1 = 4 + 3 - 1 = 6$$
$$\beta_2 = 3 + 3 - 2 = 4$$
$$\beta_3 = 2 + 3 - 3 = 2$$

So the beta-numbers are $\{6, 4, 2\}$.

Definition 4.4 (The a-Abacus)**.** An *a-abacus* consists of a vertical runners labeled $0, 1, \ldots, a-1$. Each non-negative integer n is placed on runner $n \bmod a$ at height $\lfloor n/a \rfloor$. To represent a partition, place beads at the positions corresponding to its beta-numbers.

Figure 4 shows the 3-abacus for the partition $(4, 3, 2)$.

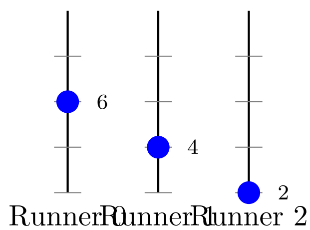


Figure 4. 3-abacus for $(4, 3, 2)$ with beta-numbers $\{6, 4, 2\}$.

The crucial insight is how rim hook removal appears on the abacus.

Proposition 4.5 (Rim Hooks and the Abacus)**.** *Removing an a-rim hook from a partition corresponds to sliding one bead up by one position on its runner. This operation is possible if and only if the position directly above the bead is empty.*

Proof. Removing an a-rim hook changes one beta-number β to $\beta - a$. On the a-abacus, β and $\beta - a$ are on the same runner (since $\beta \equiv \beta - a \pmod{a}$) but differ by one in the height coordinate $\lfloor \cdot/a \rfloor$. ∎

Theorem 4.6 (Abacus Characterization of Cores)**.** *A partition λ is an a-core if and only if, on its a-abacus, the beads on each runner are "flush"—they occupy consecutive positions starting from height 0 with no gaps.*

Proof. By Proposition 4.5, λ has no a-rim hook (equivalently, no hook of length divisible by a) if and only if no bead can slide up, if and only if there are no gaps below any bead. ∎

The abacus characterization leads to a useful reformulation in terms of hook lengths.

Lemma 4.7 (First-Column Hook Characterization)**.** *A partition λ is an a-core if and only if its set of first-column hook lengths $H = \{h_{1,1}, h_{2,1}, \ldots, h_{k,1}\}$ satisfies: whenever $n \in H$ and $n \geq a$, we have $n - a \in H$.*

Proof. The first-column hook lengths of λ are precisely the beta-numbers $\{\beta_1, \ldots, \beta_k\}$. On the a-abacus, the beads are flush (no gaps) if and only if whenever a bead appears at position $n \geq a$, there is also a bead at position $n - a$. This is exactly the closure condition on H. ∎

Definition 4.8 ((a, b)-Core)**.** A partition is an *(a, b)-core* if it is simultaneously an a-core and a b-core.

5. Anderson's Bijection

We now establish the central bijection connecting (a, b)-Dyck paths to (a, b)-cores. The key insight is that both objects can be encoded by the same combinatorial data: a sequence describing which residue classes modulo a and modulo b are "occupied." The Dyck path encodes this through its boundary word, while the core encodes it through its abacus configuration.

To construct the bijection, we work with the lattice $\mathbb{Z}^2$ and assign labels to each point. These labels will serve as the bridge between the geometric world of paths and the algebraic world of partitions.

Definition 5.1 (Residue Labeling)**.** For integers a, b with $\gcd(a, b) = 1$, define the *label* of lattice point (x, y) to be

$$\ell(x, y) = bx - ay.$$

The labels have several important properties that will be essential for our construction.

Lemma 5.2 (Properties of Labels)**.** *The labeling $\ell(x, y) = bx - ay$ satisfies:*

1. $\ell(x + 1, y) = \ell(x, y) + b$ *(moving East increases label by b)*
2. $\ell(x, y + 1) = \ell(x, y) - a$ *(moving North decreases label by a)*
3. *The point (x, y) lies strictly below the diagonal $y = \frac{b}{a}x$ if and only if $\ell(x, y) > 0$*
4. *Since* $\gcd(a, b) = 1$*, every integer appears exactly once in the region $0 \le x < a$, $0 \le y < b$*

Proof. Properties (1) and (2) follow directly from the definition. For (3), the point (x, y) is strictly below $y = \frac{b}{a}x$ iff $y < \frac{b}{a}x$ iff $ay < bx$ iff $bx - ay > 0$. For (4), the labels in this region form the set $\{bx - ay : 0 \le x < a, 0 \le y < b\}$. Since $\gcd(a, b) = 1$, the elements $b \cdot 0, b \cdot 1, \ldots, b \cdot (a - 1)$ are distinct modulo a. Combined with shifts by multiples of a, this covers all residue classes, giving ab distinct values. These range from $-a(b - 1)$ to $b(a - 1)$, which is exactly ab consecutive integers. ∎

With this labeling in hand, we can now describe the key object that mediates between paths and partitions. Given a Dyck path, we consider the lattice points that lie "between" the path and the diagonal, and we record their labels.

Definition 5.3 (Boundary Set of a Path)**.** For an (a, b)-Dyck path P, define the *boundary set* $B(P)$ to be the set of labels of lattice points (x, y) that satisfy:

(i) $\ell(x, y) \ge 0$ (weakly below the diagonal), and
(ii) (x, y) lies weakly above the path P.

In particular, $B(P)$ always contains the origin with label $\ell(0, 0) = 0$.

The boundary set has a crucial closure property that will ensure the resulting partition is a core.

Lemma 5.4 (Structure of the Boundary Set)**.** *For any (a, b)-Dyck path P:*

1. *$B(P)$ consists of non-negative integers, with $0 \in B(P)$ always*
2. *If $n \in B(P)$ and $n \ge a$, then $n - a \in B(P)$*
3. *If $n \in B(P)$ and $n \ge b$, then $n - b \in B(P)$*

Proof. Property (1) follows from the definition: $B(P)$ contains exactly those labels $\ell(x, y) \ge 0$, and the origin $(0, 0)$ is always weakly above any path starting at the origin.

For property (2), suppose $n = \ell(x,y) \in B(P)$ with $n \geq a$. By Lemma 5.2(2), the label $n-a = \ell(x,y+1)$ belongs to the point directly North of (x,y). Since $n-a \geq 0$, this point is on or below the diagonal. Dyck paths only move East or North, so if (x,y) is weakly above the path, then $(x,y+1)$ is also weakly above the path. Therefore $n-a \in B(P)$.

For property (3), suppose $n = \ell(x,y) \in B(P)$ with $n \geq b$. Since $n = bx - ay \geq b$ and $y \geq 0$, we must have $x \geq 1$. The point $(x-1,y)$ has label $\ell(x-1,y) = b(x-1) - ay = n-b \geq 0$, so it is on or below the diagonal. Since the path never moves West (only East or North), if (x,y) is weakly above the path, then $(x-1,y)$ is also weakly above the path. Therefore $n-b \in B(P)$. ■

We can now define the bijection. The idea is to use the boundary set as the first-column hook lengths of a partition.

Definition 5.5 (The Map ϕ)**.** Given an (a,b)-Dyck path P, define $\phi(P)$ to be the partition whose first-column hook lengths are exactly the elements of $B(P)$.

More explicitly, if $B(P) = \{h_1 > h_2 > \cdots > h_k \geq 0\}$, then $\phi(P)$ is the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ where $\lambda_i = h_i - k + i$.

We must verify that this construction produces valid partitions, and that these partitions are indeed cores.

Lemma 5.6 (Well-Definedness)**.** *For any (a,b)-Dyck path P, the map $\phi(P)$ produces a valid partition.*

Proof. We must show that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. The first-column hook lengths of any partition satisfy $h_i = \lambda_i + (k-i)$, so $\lambda_i = h_i - k + i$. Since $h_1 > h_2 > \cdots > h_k$ are strictly decreasing, we have

$$\lambda_i - \lambda_{i+1} = (h_i - k + i) - (h_{i+1} - k + i + 1) = h_i - h_{i+1} - 1 \geq 0$$

since $h_i > h_{i+1}$ implies $h_i - h_{i+1} \geq 1$. Also, $\lambda_k = h_k - k + k = h_k \geq 0$, and if $h_k = 0$, then $\lambda_k = 0$ which means we have one fewer row. ■

Lemma 5.7 ($\phi(P)$ is an (a,b)-Core)**.** *For any (a,b)-Dyck path P, the partition $\phi(P)$ is an (a,b)-core.*

Proof. We use the characterization from Lemma 4.7: λ is an a-core iff its first-column hook lengths form a set S such that $n \in S$ and $n \geq a$ implies $n - a \in S$.

By Lemma 5.4(2) and (3), the boundary set $B(P)$ has exactly this closure property for both a and b. Since the first-column hook lengths of $\phi(P)$ are precisely the elements of $B(P)$, the partition $\phi(P)$ is both an a-core and a b-core. ■

Having shown that ϕ maps paths to cores, we now verify that it is a bijection.

Lemma 5.8 (Injectivity)**.** *The map ϕ is injective: distinct paths give distinct cores.*

Proof. Two different (a,b)-Dyck paths define different regions below the diagonal, hence different boundary sets $B(P)$. Since $\phi(P)$ is determined by $B(P)$, distinct paths give distinct partitions. ■

Lemma 5.9 (Surjectivity)**.** *Every (a,b)-core arises as $\phi(P)$ for some (a,b)-Dyck path P.*

Proof. Let λ be an (a,b)-core with first-column hook lengths $H = \{h_1 > h_2 > \cdots > h_k\}$. Since λ is an a-core and a b-core, the set H is closed under subtracting a or b (when the result is non-negative).

We construct a path P as follows. Consider the lattice points with labels in H. The "upper boundary" of these points (the minimal set of points above which all of H lies) defines a lattice path from $(0,0)$ to (a,b).

The closure properties of H ensure this path stays below the diagonal: if the path ever crossed above, there would be a point with a negative label in H, which is impossible. Thus P is an (a,b)-Dyck path with $\phi(P) = \lambda$. ■

The final piece of the bijection is the remarkable identity that makes it useful for computing statistics: the area under a path equals the size of the corresponding partition.

Definition 5.10 (Area of a Path)**.** The *area* $\text{Area}(P)$ of an (a,b)-Dyck path is the number of unit squares in the region $\{(x,y) : 0 \le x \le a, 0 \le y \le b\}$ that lie strictly between the path and the line $y = \frac{b}{a}x$.

Lemma 5.11 (Area-Size Identity)**.** *For any (a,b)-Dyck path P, we have* $\text{Area}(P) = |\phi(P)|$.

Proof. We sketch the key idea; a complete proof appears in Anderson [1].

Anderson shows that the non-negative labels in $B(P)$, together with their closure properties under subtracting a and b, encode a partition whose Young diagram fills exactly the region between the path and the diagonal. Each cell of the Young diagram corresponds to one unit square in this region, establishing $|\phi(P)| = \text{Area}(P)$. ■

We can now state the complete result.

Theorem 5.12 (Anderson's Bijection)**.** *The map ϕ is a bijection from (a,b)-Dyck paths to (a,b)-cores satisfying*

$$\text{Area}(P) = |\phi(P)|.$$

Proof. Injectivity follows from Lemma 5.8, surjectivity from Lemma 5.9, and the area-size identity from Lemma 5.11. ■

6. Composition Sums

To compute the average area, we need to understand how area depends on the structure of a path. We decompose paths by their "valleys."

Definition 6.1 (Valley)**.** A *valley* in a lattice path is an East step immediately followed by a North step.

A path with k valleys consists of k East-runs of lengths $r_1, \ldots, r_k$ (summing to a) alternating with k North-runs of lengths $s_1, \ldots, s_k$ (summing to b).

Definition 6.2 (Composition)**.** A *composition* of n into k parts is an ordered tuple $(t_1, \ldots, t_k)$ of positive integers with $\sum_{i=1}^{k} t_i = n$.

Lemma 6.3 (Composition Count)**.** *The number of compositions of n into k parts is $\binom{n-1}{k-1}$.*

Proof. Imagine n objects in a row with $n-1$ gaps between them. Choosing $k-1$ gaps as dividers partitions the objects into k nonempty groups. ■

Lemma 6.4 (Rational Narayana Numbers)**.** *The number of (a, b)-Dyck paths with exactly k valleys, for $1 \le k \le \min(a, b)$, is*

$$N_{a,b}(k) = \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}. \tag{6.1}$$

Proof sketch. This follows from a refined version of the cycle lemma that tracks the number of valleys. We omit the details; see [2] for a complete proof. ∎

Lemma 6.5 (Composition Sum Identity)**.** *For any function f and positive integers n, k with $k \le n$:*

$$\sum_{\substack{(t_1,\ldots,t_k)\\ \sum t_i = n}} \sum_{i=1}^{k} f(t_i) = k \sum_{t=1}^{n-k+1} f(t)\binom{n-t-1}{k-2}. \tag{6.2}$$

Proof. We count pairs (composition, index). For each pair, one part t_i takes value t. By symmetry over the k positions, we get a factor of k. For a fixed value t, the remaining $k-1$ parts must sum to $n-t$, which has $\binom{n-t-1}{k-2}$ compositions. ∎

7. The Average Size Calculation

We now sketch the derivation of the main result. The full calculation involves generating function techniques and binomial identities; we present the key ideas and refer to Armstrong, Hanusa, and Jones [2] for complete details.

The natural approach is to express area in terms of a path's structure. For a path with k valleys, let $(r_1, \ldots, r_k)$ be the East-run lengths and $(s_1, \ldots, s_k)$ be the North-run lengths. The area can be written as

$$\text{Area} = \sum_{i=1}^{k} h_i \cdot r_i \tag{7.1}$$

where $h_i = s_1 + s_2 + \cdots + s_{i-1}$ is the height at which East-run i begins.

A heuristic calculation. If we *pretend* that for a uniform random Dyck path with k valleys, the run lengths $(r_1, \ldots, r_k)$ and $(s_1, \ldots, s_k)$ behave like uniform random compositions (which they do not, due to the Dyck constraint), we would have:

- $\mathbb{E}[r_i] \approx \frac{a}{k}$ for each i
- $\mathbb{E}[s_j] \approx \frac{b}{k}$ for each j
- $\mathbb{E}[h_i] \approx \frac{(i-1)b}{k}$

This would give an expected area contribution of $\frac{(i-1)ab}{k^2}$ from index i, summing to $\frac{ab(k-1)}{2k}$ for a path with k valleys.

While this heuristic is not rigorous (the Dyck constraint biases the distribution of runs), it correctly predicts the form of the answer. The rigorous calculation uses generating functions.

The rigorous approach. Armstrong, Hanusa, and Jones [2] evaluate the area generating function

$$\sum_{P} q^{\text{Area}(P)} = \sum_{k=1}^{\min(a,b)} N_{a,b}(k) \cdot [\text{contribution from paths with } k \text{ valleys}]$$

using q-binomial identities and properties of the rational Narayana numbers $N_{a,b}(k) = \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}$.

Setting $q = 1$ and differentiating extracts the expected area. After substantial algebraic manipulation involving Vandermonde's convolution, one obtains:

$$\sum_P \text{Area}(P) = \frac{(a-1)(b-1)(a+b+1)}{24} \cdot \text{Cat}(a,b). \tag{7.2}$$

Dividing by the total count $\text{Cat}(a,b)$ gives the average:

$$\mathbb{E}[\text{Area}] = \frac{(a-1)(b-1)(a+b+1)}{24}. \tag{7.3}$$

By Anderson's bijection (Theorem 5.12), this equals the expected size of an (a,b)-core.

Example. For $(a,b) = (3,4)$, the formula gives $\frac{2\cdot 3\cdot 8}{24} = 2$. The five $(3,4)$-cores are: $\emptyset$ (size 0), (1) (size 1), (2) (size 2), $(1,1)$ (size 2), and $(2,1,1,1)$ (size 5). Their total size is $0+1+2+2+5 = 10$, and $10/5 = 2$.

8. Conclusion

Our journey through the theory of (a,b)-core partitions has revealed remarkable connections spanning lattice path enumeration, the representation theory of partitions via the abacus, and composition sum techniques. What began as a question about the average size of partitions avoiding certain hook lengths has led us to the elegant formula $(a-1)(b-1)(a+b+1)/24$.

The cycle lemma stands as the central counting tool, providing not only the enumeration of (a,b)-Dyck paths but also the foundation for Anderson's bijection. Through James' abacus, we have seen how the seemingly abstract concept of core partitions admits a visual representation that makes their structure transparent.

The theory naturally extends in several directions. The study of simultaneous $(a_1,\ldots,a_k)$-cores for more than two values leads to deeper questions about the structure of the core poset. The fluctuations of core sizes around their expected value connect to Tracy-Widom distributions and random matrix theory. Higher moments and q-analogs remain active research directions.

9. Acknowledgements

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