PÓLYA'S ENUMERATION THEOREM

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1. Introduction

The problem of counting colored necklaces is a simple yet revealing example in combinatorics: although at first glance it seems enough to note that there are m^n ways to assign m colors to a necklace with n beads, this number completely ignores the geometric symmetries of the object. Two configurations that differ only by a rotation (or reflection, if these are allowed) represent, from a geometric standpoint, the same necklace. Consequently, the count m^n does not appear to be a very satisfactory answer, since it massively overcounts the genuinely distinct configurations.

To correct this overcounting we introduce an equivalence relation that identifies symmetric colorings. Such relations, induced by group actions, allow us to systematically organize the configurations into classes, each of which captures an essential pattern. Burnside's Lemma provides the fundamental bridge between symmetries and the counting of these classes.

Pólya's Enumeration Theorem extends these ideas by allowing us to simultaneously record the cycle structure of each permutation and the exact distribution of colors on the necklace.

In Section 2 we review preliminary notions regarding equivalence relations, permutations, and group actions, restricting ourselves to the minimal framework required. In Section 3 we present Burnside's Lemma and its cycle–index version, applying them to enumerate necklaces under symmetries. Section 4 is devoted to Pólya's Enumeration Theorem, including explicit examples.

2. Preliminaries

2.1. Equivalence Relations.

Definition 2.1. A relation R on a set S is a subset of $S \times S$. We say that two elements $a, b \in S$ are related if $(a, b) \in R$, and we denote this by $a \sim b$.

Definition 2.2. A relation R on a set S is called an equivalence relation if the following conditions hold:

- For all $x \in S$, $x \sim x$ (reflexivity).
- For all $x, y \in S$, if $x \sim y$ then $y \sim x$ (symmetry).
- For all $x, y, z \in S$, if $x \sim y$ and $y \sim z$ then $x \sim z$ (transitivity).

Definition 2.3. Given an equivalence relation R on S, the *equivalence class of x induced by R* is the set

$$cl(x) = \{y \in S : x \sim y\}.$$

Proposition 2.4. If R is an equivalence relation on S, then every element of S belongs to exactly one equivalence class.

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Proof. Since R is reflexive, we know that $x \sim x$ for all $x \in S$, so $x \in cl(x)$. Now suppose there exists some $x \in S$ such that $x \in cl(y)$ and $x \in cl(z)$, where y and z are elements of S. Since R is symmetric, we have $y \sim x$ and $x \sim z$. Since R is transitive, we obtain $y \sim z$, so $cl(y) = cl(z)$. ■

2.2. **Permutations.**

Recall that S_n denotes the set of all bijections from $[n]$ to $[n]$. Since function composition is closed in S_n, any permutation of $[n]$ can be composed with another, and therefore S_n may be viewed as the set of all permutations of $[n]$. We call the elements of S_n *permutations*.

Definition 2.5. Let $\pi \in S_n$. For $i \in [n]$ we define the sequence

$$i,\ \pi(i),\ \pi^2(i), \dots$$

which must eventually return to i since π is a bijection. The set of elements visited in this process forms a *cycle* of the permutation π. If a cycle contains the elements $a_1, a_2, \dots, a_k$, we write it using the notation

$$(a_1\ a_2\ \cdots\ a_k).$$

These observations lead to the following fact.

Proposition 2.6. Every permutation $\pi \in S_n$ can be written as a product of disjoint cycles, and this expression is unique up to the order of the cycles.

Definition 2.7. Let $\pi \in S_n$ and suppose that its disjoint-cycle decomposition contains exactly k cycles. We define this number as $\text{cyc}(\pi) = k$.

2.3. **Group theory.**

Definition 2.8. A group is a set G equipped with a binary operation $*$ satisfying:

- For all $x, y \in G$, one has $x * y \in G$.
- For all $x, y, z \in G$, $(x * y) * z = x * (y * z)$.
- There exists an element $e \in G$ such that $x * e = e * x = x$ for all $x \in G$.
- For each $x \in G$ there exists an element $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e$.

Definition 2.9. Let G be a group and let $H \subseteq G$. If H, with the operation $*$ restricted, forms a group in its own right, we say that H is a *subgroup* of G.

Definition 2.10. We say that a group G is generated by a set $S = \{g_1, \dots, g_n\}$ if every element of G can be expressed as a product of elements of S. When this happens, we say that $g_1, \dots, g_n$ are the *generators* of G, and we write

$$G = \langle g_1, \dots, g_n \rangle.$$

Definition 2.11. We say that a group G acts on a set X if G is a group of permutations of X. That is, for every $g \in G$ and every $x \in X$ we have $g(x) \in X$.

Although this definition is not the usual one, it will serve for our purposes.

Definition 2.12. Let G be a group acting on a set X. For each $g \in G$, we define

$$\text{Inv}(g) = \{x \in X : g(x) = x\},$$

the set of *invariants* of g.

Definition 2.13. Let G be a group acting on a set X. For each $x \in X$, we define

$$\text{st}(x) = \{g \in G : g(x) = x\},$$

the *stabilizer* of x.

We now describe two fundamental subgroups of the symmetric group S_n, which model the symmetries of a necklace with n positions.

Definition 2.14. Let $n \geq 1$. We define the permutation

$$r = (1\ 2\ 3\ \cdots\ n) \in S_n,$$

which sends i to $i+1$ for $1 \leq i < n$, and sends n to 1. The *cyclic group of order* n, denoted C_n, is defined as

$$C_n = \langle r \rangle = \{\, r^k : 0 \leq k < n \,\}.$$

Definition 2.15. Let $n \geq 1$. In addition to r as above, we define the permutation $s \in S_n$ by

$$s(i) = n + 1 - i \qquad (1 \leq i \leq n),$$

which corresponds to the reflection that reverses the order of the positions. The *dihedral group of order* $2n$, denoted D_n, is defined as

$$D_n = \langle r, s \rangle.$$

For a broader treatment of abstract algebra, we refer the reader to Fraleigh [1] or Herstein [2].

3. Burnside's Lemma

When a group acts on a set X, each element of X may be transformed into other elements via the action. We say that x and y are equivalent if there exists $g \in G$ such that $g(x) = y$. This relation is reflexive, symmetric, and transitive, and therefore induces an equivalence relation. The resulting classes are exactly the subsets of the form

$$\mathrm{cl}(x) = \{g(x) : g \in G\}.$$

Theorem 3.1 (Burnside's Lemma)**.** Let G be a group acting on a finite set X. The number of equivalence classes induced by the action is

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Inv}(g)|.$$

Proof. Let N be the number of pairs $(g, x) \in G \times X$ such that $g(x) = x$. We count N in two ways.

On one hand, for fixed g, the number of x such that $g(x) = x$ is $|\mathrm{Inv}(g)|$, so

$$N = \sum_{g \in G} |\mathrm{Inv}(g)|.$$

On the other hand, we group the pairs (g, x) by equivalence classes. Let $x \in X$ and consider its class $\mathrm{cl}(x)$. If $y \in \mathrm{cl}(x)$, then $|\operatorname{st}(y)| = |\operatorname{st}(x)|$ because the stabilizers of elements in the same class have equal size.

We now use the Orbit–Stabilizer Theorem, which states that for every $x \in X$

$$|G| = |\mathrm{cl}(x)| \cdot |\operatorname{st}(x)|.$$

This implies that the number of pairs (g, y) with $y \in \mathrm{cl}(x)$ and $g(y) = y$ is

$$|\mathrm{cl}(x)| \cdot |\operatorname{st}(x)| = |G|.$$

If there are t equivalence classes, we obtain

$$N = t \cdot |G|.$$

Comparing with the first expression for N, we conclude

$$t = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Inv}(g)|.$$

■

Definition 3.2. Let S be a finite set and C a set of colors. A *coloring* is a function

$$f : S \to C.$$

For a necklace with n positions, there are $|C|^n$ possible colorings.

Proposition 3.3. Let π be a permutation of the positions of a necklace. If π has $\mathrm{cyc}(\pi)$ disjoint cycles, then

$$|\mathrm{Inv}(\pi)| = |C|^{\mathrm{cyc}(\pi)}.$$

Proof. Each cycle requires that the positions it contains receive the same color in order for the coloring to be fixed by π. Since the cycles are disjoint, each one may freely choose one of the $|C|$ colors. Therefore the total number of invariant colorings is $|C|^{\mathrm{cyc}(\pi)}$. ■

Thus, we may reformulate the result as follows.

Theorem 3.4 (Cycle–Index Version of Burnside's Lemma)**.** Let G act by permutations on a set of n positions. If there are m available colors, the number of equivalence classes of colorings under the action of G is

$$\frac{1}{|G|} \sum_{\pi \in G} m^{\mathrm{cyc}(\pi)}.$$

Example. Count the necklaces with 6 beads and m colors, identifying those obtained by rotations. Here $G = C_6 = \langle r \rangle$, where

$$r = (1\ 2\ 3\ 4\ 5\ 6).$$

The elements of C_6 are

$$r^0,\ r^1,\ r^2,\ r^3,\ r^4,\ r^5.$$

We decompose each into cycles:

$$r^0 = (1)(2)(3)(4)(5)(6), \quad \mathrm{cyc}(r^0) = 6,$$
$$r^1 = (1\ 2\ 3\ 4\ 5\ 6), \quad \mathrm{cyc}(r^1) = 1,$$
$$r^2 = (1\ 3\ 5)(2\ 4\ 6), \quad \mathrm{cyc}(r^2) = 2,$$
$$r^3 = (1\ 4)(2\ 5)(3\ 6), \quad \mathrm{cyc}(r^3) = 3,$$
$$r^4 = r^2, \quad \mathrm{cyc}(r^4) = 2,$$
$$r^5 = r^1, \quad \mathrm{cyc}(r^5) = 1.$$

Applying Burnside:

$$\frac{1}{6}\big(m^6 + m^1 + m^2 + m^3 + m^2 + m^1\big) = \frac{1}{6}(m^6 + m^3 + 2m^2 + 2m).$$

This is the total number of distinct 6-bead necklaces up to rotation.

Example. Now count necklaces with 4 beads and m colors, identifying those obtained by rotations and reflections. Here $G = D_4 = \langle r, s\rangle$, where

$$r = (1\ 2\ 3\ 4)$$

and s is a reflection. In 4 positions there are two types of reflections: those that fix two points and those that exchange two pairs.

The elements of D_4 are:

$$r^0,\ r^1,\ r^2,\ r^3,\ s,\ sr,\ sr^2,\ sr^3.$$

Cycle decompositions:

$$\begin{aligned} r^0 &= (1)(2)(3)(4), \quad \mathrm{cyc}(r^0) = 4,\\ r^1 &= (1\ 2\ 3\ 4), \quad \mathrm{cyc}(r^1) = 1,\\ r^2 &= (1\ 3)(2\ 4), \quad \mathrm{cyc}(r^2) = 2,\\ r^3 &= (1\ 4\ 3\ 2), \quad \mathrm{cyc}(r^3) = 1.\end{aligned}$$

For the reflections we take representatives:

$$\begin{aligned} s &= (2\ 4), \quad \mathrm{cyc}(s) = 3,\\ sr &= (1\ 2)(3\ 4), \quad \mathrm{cyc}(sr) = 2,\\ sr^2 &= s, \quad \mathrm{cyc}(sr^2) = 3,\\ sr^3 &= sr, \quad \mathrm{cyc}(sr^3) = 2.\end{aligned}$$

Applying the formula:

$$\frac{1}{8}\big(m^4 + m^1 + m^2 + m^1 + m^3 + m^2 + m^3 + m^2\big).$$

Grouping:

$$\frac{1}{8}\big(m^4 + 2m^3 + 3m^2 + 2m\big).$$

This is the number of distinct 4-bead necklaces under rotations and reflections.

4. Pólya's Enumeration Theorem

In the final version we present, we wish to count necklaces that use specific quantities of each color. To do this we must refine the method: rather than merely counting how many configurations exist, we want to record how the colors are distributed. This is where Pólya's Enumeration Theorem enters, converting Burnside's Lemma into a method capable of simultaneously recording the cycle structure and the precise assignment of colors.

Recall that a cycle of length i in π requires all its positions to have the same color. Instead of simply counting how many color choices there are, we introduce variables that represent the colors and let the cycle structure determine the power with which they appear. We do this using variables $x_1, \ldots, x_n$, where x_i expresses the "contribution" of a cycle of length i.

Definition 4.1. Let G be a subgroup of S_n. If $\pi \in G$ has exactly $k_i(\pi)$ cycles of length i in its disjoint-cycle decomposition, its *cycle-index monomial* is

$$\prod_{i=1}^{n} x_i^{k_i(\pi)}.$$

Definition 4.2. The *cycle-index polynomial* of G is

$$Z_G(x_1, \ldots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} \prod_{i=1}^{n} x_i^{k_i(\pi)}.$$

This polynomial collects relevant information about the group: for each π, the corresponding monomial encodes how many cycles of each length it contains.

To incorporate colors, we assign to each color c a variable y_c. A cycle of length i must receive a single color, so its contribution is y_c^i.

The total contribution is obtained by substituting

$$x_i = \sum_{c \in C} y_c^i$$

into the cycle-index polynomial Z_G. The expansion of the result contains all non-equivalent patterns, and the powers of the variables y_c indicate how often each color appears.

Theorem 4.3 (Pólya's Enumeration Theorem)**.** Let $G \leq S_n$ act on n positions. Let C be a set of colors and assign to each color $c \in C$ a variable y_c. If in the cycle-index polynomial Z_G we perform the substitution

$$x_i = \sum_{c \in C} y_c^i,$$

then the resulting polynomial encodes all distinct colorings under the action of G. In particular, the coefficient of the monomial

$$\prod_{c \in C} y_c^{a_c}$$

is the number of equivalence classes of colorings that use exactly a_c positions of color c.

Example. Count necklaces with 5 beads using three colors R, G, B, distinguishing configurations only up to rotation.

The elements of $C_5 = \langle r \rangle$ are

$$r^0, \ r^1, \ r^2, \ r^3, \ r^4$$

with decompositions:

$$r^0 = (1)(2)(3)(4)(5), \quad \text{cyc}(r^0) = 5,$$
$$r^1 = (1\ 2\ 3\ 4\ 5), \quad \text{cyc}(r^1) = 1,$$

and for r^2, r^3, r^4 the same occurs as for r^1, since all are cycles of length 5, hence

$$\text{cyc}(r^1) = \text{cyc}(r^2) = \text{cyc}(r^3) = \text{cyc}(r^4) = 1.$$

We write their cycle-index monomials:

$$r^0 \longmapsto x_1^5, \qquad r^k \longmapsto x_5 \quad (k = 1, 2, 3, 4).$$

The cycle-index polynomial is

$$Z_{C_5}(x_1, \ldots, x_5) = \frac{1}{5}\big(x_1^5 + 4x_5\big).$$

We substitute:

$$x_1 = R + G + B, \qquad x_5 = R^5 + G^5 + B^5.$$

We obtain

$$Z_{C_5}(R, G, B) = \frac{1}{5}\Big((R + G + B)^5 + 4(R^5 + G^5 + B^5)\Big).$$

Now, if we want to count necklaces with exactly two red beads, one green bead, and two blue beads, we look for the coefficient of the monomial

$$R^2G^1B^2.$$

The expansion of the first term $(R+G+B)^5$ contains all monomials of degree 5; the second term contributes only monomials of the form R^5, G^5, B^5. Therefore the desired coefficient is

$$\frac{1}{5} \cdot \binom{5}{2,1,2}.$$

Thus, the number of necklaces with exactly two red beads, one green bead, and two blue beads (up to rotation) is 6.

References

[1] John B. Fraleigh, *A First Course in Abstract Algebra*, Addison–Wesley.
[2] I. N. Herstein, *Topics in Algebra*, Wiley.

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