You Might Be Able to Divide By n

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June 28, 2000

1 A Tangled Mess

Let n be a positive integer. What does it mean to divide by n?

The answer is more philosophical than it seems. Of course, given an integer n, there are many possible interpretations of multiplication by n, and consequently many corresponding notions of division n.

At first glance, the question sounds trivial. Given an integer n, we know exactly what it means to multiply by n, and so we expect the notion of "division by n" to follow from the familiar laws of arithmetic. But if we take a step back and ask where these laws come from, the situation becomes more subtle. Rather than taking the axioms of arithmetic as our starting point, we may instead try to reconstruct arithmetic combinatorially, directly from the behavior of sets and bijections.

This perspective leads us to the central problem of combinatorial division. For a set A, define

$$A \times n \ = \ A \times \{0, 1, \ldots, n-1\}.$$

Thus "multiplication by n" is interpreted as forming n disjoint labeled copies of A. Now suppose we are given sets A and B, together with a bijection

$$f : A \times n \longrightarrow B \times n.$$

Intuitively, if $A \times n$ and $B \times n$ are in bijection, then A and B "should be" in bijection as well [1]. The question is whether this intuition can be made explicit:

$$\textit{Given } f, \textit{ can we construct an explicit bijection } g : A \longrightarrow B?$$

Unfortunately, complications arise from the *tangling* nature of f. By this we mean the following: although $A \times n$ is formally the disjoint union of n labeled copies of A, the bijection

$$f : A \times n \to B \times n$$

[1]This question can be posed instead for injections, and the case of bijections follows from the Cantor-Bernstein theorem. There are a few intricacies in the general injective case, which are not covered in this exposition. However, the interested reader is encouraged to read "Division by four".

need not respect this product structure in any way. A point (a, i) in the i-th copy of A may be sent to a point (b, j) lying in the *different* j-th copy of B, and as a varies, the labels i and j may interact in highly nontrivial ways. In other words, f may freely shuffle, permute, interlace, and cycle the n fibers of A, producing a mapping whose action on the underlying sets A and B becomes deeply obscured.

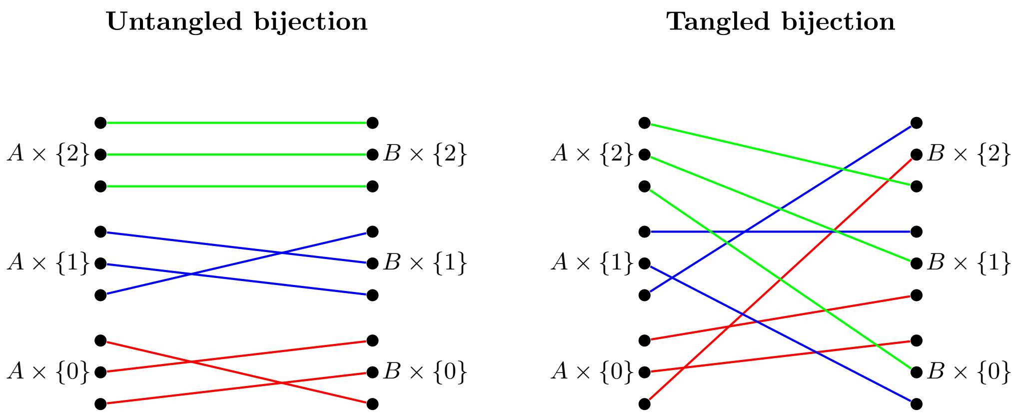

Figure 1: On the right, a tangled bijection from $A \times 3 \to B \times 3$. Extracting a bijection from $A \to B$ is nontrivial. On the left, an untangled bijection, which which one can easily extract (in fact, three, not necessarily distinct) bijections from $A \to B$. Note that an untangled bijection from $A \times 3 \to B \times 3$ is not necessarily the identity function.

To illustrate, think of each fiber $A \times \{i\}$ as a colored strand. Although multiplication by n arranges these n strands neatly in parallel, the bijection f may weave them together into a single intricate braid. A point may travel from the red copy of A into the blue copy of B, then on its inverse path reappear in the green copy of A, and so on. Because f is a bijection only on the product level, it need not leave any canonical trace of how elements of A should correspond to elements of B individually.

Thus, the fundamental issue when attempting to divide by n is untangling the structure of f.

2 Pan-Galactic Division

Theorem 2.1. *Let $n \geq 1$ be a fixed integer. If there exists a bijection*

$$f : \; n \times A \longrightarrow n \times B,$$

then there exists a bijection

$$g : \; A \longrightarrow B.$$

Equivalently, $n \times A \cong n \times B$ *implies* $A \cong B$.

Proof. We simply provide the constuction of the Pan-Galactic Division Algorithm.

Setup and notation. Index the suits by $\{0, 1, \ldots, n-1\}$. Think of $n \times B$ as a deck of cards: for each $b \in B$ and suit s there is a card (s, b). Think of $n \times A$ as the set of rack spots: for each $a \in A$ and suit s there is a spot (s, a). The bijection f places exactly one card into each spot.

Pass For each rack $a \in A$, label the spots $0, 1, \ldots, n-1$; call spot 0 the *home* spot for suit 0. A card of suit 0 in spot 0 is *good*; a card of suit 0 in some other spot is *bad.*

Shipshaping consists of repeated synchronous rounds of two moves:

1. **Shape Up.** In every rack with at least one bad suit-0 card and no good suit-0 card, move the leftmost bad suit-0 card into spot 0. Perform all such moves simultaneously.
2. **Ship Out.** For every remaining bad suit-0 card (i.e., a suit-0 card in some spot $t > 0$ in a rack already containing a good suit-0 card), swap it with the card whose suit equals t and whose rank equals that of the good suit-0 card in the same rack. Perform all such swaps simultaneously.

We record the following two observations:

- Each swap is a bijective rearrangement of cards among the racks; distinct spots remain occupied by distinct cards, so injectivity (and bijectivity) is preserved.
- Bad suit-0 cards never move leftwards; they either become good or move to a higher-indexed spot. Hence, after finitely many rounds, no bad suit-0 cards remain outside their home spots.

Consequently, after the first pass all spots $1, \ldots, n-1$ contain no suit-0 cards. Restricting to suits $\{1, \ldots, n-1\}$ yields a bijection

$$f_1 : \ (n-1) \times A \longrightarrow (n-1) \times B.$$

Repeating the Shipshaping procedure for suits $1, 2, \ldots, n-1$, one eventually produces a bijection

$$f_{n-1} : \ 1 \times A \longrightarrow 1 \times B,$$

i.e., a bijection $g : A \to B$, as required.

Remark 2.1. Shipshaping is synchronous and deterministic: at each round the leftmost bad card in Shape Up and the prescribed target in Ship Out are defined canonically from the initial bijection. Thus the resulting bijection g is canonical.

□

Intuitively, Pan–Galactic Division "untangles" the original bijection $f : n \times A \to n \times B$ by systematically organizing cards of each suit into their home spots. Each Shipshaping pass isolates one suit, moving its cards into canonical positions without collisions, and reduces the problem to fewer suits. Repeating this process recursively separates all suits, producing a well-defined bijection $g : A \to B$. In this sense, the algorithm decomposes the complex bijection on the product set into a simpler bijection between the underlying sets A and B.

3 A Messy Sock Drawer

The Pan-galactic division algorithm certainly achieves the goal of constructing a conincal bijection $A \to B$ from $A \times n \to B \times n$. However, this definition of multiplication subtly relies on an ordering, that we explain below. This leads to a distinction between what we may refer to as *shoe division* and *sock division*.

An Eccentric Millionaire of Bertrand Russel

Consider a millionaire with an eccentric habit: every morning he buys one pair of shoes and one pair of socks. After infinitely many days, he owns countably many pairs of each. He then asks his butler to display *one* shoe from each pair. The butler asks how to choose, and the millionaire replies, "Always take the left shoe." This instruction works because every pair of shoes is *ordered*: there is a built-in distinction between left and right.

The next day, the millionaire asks his butler to display one sock from each pair. But the butler again asks: "Which sock should I pick from each pair?" Here the millionaire is stymied. Since the two socks in a pair are *indistinguishable*, there is no rule he can give that selects exactly one sock from each pair. This illustrates a basic fact of set theory without the Axiom of Choice: given a sequence of 2-element *unordered* sets, one cannot in general choose an element from each set.

This story motivates the distinction between dividing by *ordered* n-sets (shoes) and *unordered* n-sets (socks).

Shoe Division vs. Sock Division

In the Pan–Galactic setting we have been using so far, the product

$$A \times n \;=\; A \times \{0, 1, \dots, n-1\}$$

consists of *ordered* fibers. These behave like pairs of shoes: each fiber carries a canonical internal ordering inherited from $\{0, \dots, n-1\}$. Division by n in this context is *shoe division*, and it is provable in ZF.

By contrast, *sock division* replaces each ordered n-tuple with an arbitrary *unordered* n-element set. Formally, one considers two collections of disjoint n-element sets,

$$\{X_a\}_{a\in A}, \qquad \{Y_b\}_{b\in B},$$

and asks:

If $\bigcup_a X_a$ and $\bigcup_b Y_b$ are in bijection,
must A and B also be in bijection?

This is the analogue of "cancelling the factor of n" when the fibers have *no* internal labeling. As we will show, this turns out to be impossible.

You Can't Organize Your Sock Drawer

Theorem 3.1. *For any $n > 1$, sock division by n is not provable in ZF.*

Proof. Our proof follows Lutz. We show that sock division by 2 implies the existence of a choice function for a countable family of 2-element sets, contradicting ZF. Since the case $n > 2$ may be reduced to $n = 2$ by replacing the socks with n-fold "octopus socks," the theorem follows.

Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of pairwise disjoint 2-element sets. Write $A_i = \{x_i, y_i\}$. ZF cannot prove that there exists a choice function $c(i) \in A_i$ for all i.

Assume, for contradiction, that sock division by 2 holds. For each i, form the set

$$A_i \times \{0, 1\}.$$

This is a 4-element set naturally visualized as a 2×2 grid. There are two canonical ways to decompose it into disjoint 2-element subsets:

(1) **Row decomposition:**

$$R_{x_i} = \{(x_i, 0), (x_i, 1)\}, \qquad R_{y_i} = \{(y_i, 0), (y_i, 1)\}.$$

For each i we obtain two rows, indexed by the *individual socks* x_i and y_i.

(2) **Column decomposition:**

$$C_{i,0} = \{(x_i, 0), (y_i, 0)\}, \qquad C_{i,1} = \{(x_i, 1), (y_i, 1)\}.$$

For each i we obtain two columns, indexed by the pair index i together with 0 or 1.

Perform this construction for every $i \in \mathbb{N}$. We obtain two collections of disjoint 2-element sets:

$$\mathcal{R} = \{R_x : x \in \bigcup_i A_i\}, \qquad \mathcal{C} = \{C_{i,b} : (i,b) \in \mathbb{N} \times \{0,1\}\}.$$

Crucially,

$$\bigcup \mathcal{R} \ = \ \bigcup_{i \in \mathbb{N}} (A_i \times \{0,1\}) \ = \ \bigcup \mathcal{C}.$$

Thus the unions of $\mathcal{R}$ and $\mathcal{C}$ coincide.

By sock division, the indexing sets must therefore satisfy

$$\bigcup_{i \in \mathbb{N}} A_i \ \cong \ \mathbb{N} \times \{0,1\}.$$

Let

$$f : \bigcup_{i \in \mathbb{N}} A_i \longrightarrow \mathbb{N} \times \{0,1\}$$

be such a bijection.

We now use f to define a choice function for the sequence $\{A_i\}$. For any i, the elements of A_i are x_i and y_i, and $f(x_i) \neq f(y_i)$. Using the lexicographic order on $\mathbb{N} \times \{0,1\}$, define

$$c(i) = \begin{cases} x_i & \text{if } f(x_i) <_{\text{lex}} f(y_i), \\ y_i & \text{otherwise.} \end{cases}$$

This produces a well-defined function c with $c(i) \in A_i$ for all i. Thus we have constructed a choice function for the family $\{A_i\}$, contradicting the fact that ZF cannot prove the existence of such a function.

Therefore sock division by 2 is impossible in ZF, and the same holds for all $n > 1$. □

In short, shoe division works because each fiber contains an internal linear order. Sock division fails because unordered n-element fibers do not carry enough structure to allow untangling.