Q-ANALOGUES

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Abstract. We discuss the basics of q-analogues. We discuss the q-integers, q-factorials, and q-binomials, their connections to integer partitions, permutation statistics and finite fields. We then cover the basics of q-calculus.

1. introduction

A q-analogue is a deformation of a familiar mathematical object that depends on a parameter q and returns to its classical form when $q = 1$. Many constructions admit such a refinement. Numerical quantities such as factorials and binomial coefficients become polynomials or power series that record additional combinatorial structure, and algebraic or analytic identities acquire parallel q-versions with similar behavior.

The purpose of this paper is to outline several basic examples and to describe the relationships between them. Section 2 introduces q-integers, q-factorials, and q-binomial coefficients, and develops the identities that lead to the q-binomial theorem and the Jacobi Triple Product. Section 3 presents combinatorial interpretations in terms of permutation statistics as well as linear algebra over finite fields. Section 4 discusses the q-derivative, the q-Taylor expansion, and the two standard q-exponential functions, and relates these constructions to the product formulas of Section 2.

These examples illustrate the breadth and coherence of the subject and the way q-analogues unify ideas that arise in many areas of mathematics.

2. The q-binomial

The most important q-analogues are the q-binomial, q-factorials, and q-integers. There are numerous ways of defining these but I think the easiest is to begin by finding a q-analogue of the natural numbers. Natural numbers are almost always defined via recurrence, namely $n = 1 + n - 1$, since we are trying to find a polynomial in q it would not be unreasonable to deform this recurrence to $[n]_q = 1 + q[n-1]_q$. Given the pretty reasonable initial condition $[1]_q = 1$ we get the following definition.

Definition 2.1. For any complex number q and any natural number n the n-th q-integer denoted $[n]_q$ is defined by:

$$[n]_q = \sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}$$

Notably this final equality here allows us to define $[n]_q$ not just for natural numbers but for any complex number n.

This in turn allows us to define the q-factorial

Definition 2.2. For any natural number n define the q-factorial of n to be

$$[n]_q! = [n]_q[n-1]_1 \cdots [2]_q[1]_q$$

Which then allows us to define the q-binomial coefficients

Definition 2.3. For any natural numbers n, k define

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

This has a couple of obvious equivalent forms, in particular

$$\begin{aligned}\frac{[n]_q!}{[k]_q![n-k]_q!} &= \frac{\prod_{m=0}^{k-1}[n-m]_q}{\prod_{m=1}^{k}[m]_q} \\ &= \frac{\prod_{m=0}^{k-1}\frac{1-q^{n-m}}{1-q}}{\prod_{m=1}^{k}\frac{1-q^m}{1-q}} \\ &= \frac{\prod_{m=0}^{k-1}(1-q^{n-m})}{\prod_{m=1}^{k}(1-q^m)} \\ &= \prod_{m=1}^{k}\frac{(1-q^{n-m+1})}{1-q^m} = \prod_{m=1}^{k}\frac{(q^{n-m+1}-1)}{q^m-1}\end{aligned}$$

Famously, the binomial coefficients are given by the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Given that we defined $[n]_q$ via deforming the recurrence on the natural numbers one could reasonably ask why we didn't define $\binom{n}{k}_q$ by deforming its recurrence in a similar way. Luckily, we needn't worry, our definition of the q-binomial coefficients fulfills an analogue of the recurrence.

Lemma 2.4. *For any integers n, k and any $q \in \mathbb{C}$ we have*

$$\binom{n}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

and

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q$$

Proof. First note that

$$\frac{1-q^n}{1-q^{n-k}}\binom{n-1}{k}_q = \frac{1-q^n}{1-q^{n-k}}\prod_{m=1}^{k}\frac{(1-q^{n-m})}{1-q^m} = \prod_{m=1}^{k}\frac{(1-q^{n-m+1})}{1-q^m} = \binom{n}{k}_q$$

And

$$\frac{1-q^n}{1-q^k}\binom{n-1}{k-1}_q = \frac{1-q^n}{1-q^k}\prod_{m=1}^{k-1}\frac{(1-q^{n-m})}{1-q^m} = \prod_{m=1}^{k}\frac{(1-q^{n-m+1})}{1-q^m} = \binom{n}{k}_q$$

And so in particular

$$\frac{1-q^k}{1-q^{n-k}}\binom{n-1}{k}_q = \binom{n-1}{k-1}_q$$

As well as

$$\binom{n-1}{k}_q = \frac{1-q^{n-k}}{1-q^k}\binom{n-1}{k-1}_q$$

Because $\frac{1-q^n}{1-q^{n-k}} = q^k + \frac{1-q^k}{1-q^{n-k}}$ and $\frac{1-q^n}{1-q^k} = q^{n-k} + \frac{1-q^{n-k}}{1-q^k}$ we thus have

$$\binom{n}{k}_q = (q^k + \frac{1-q^k}{1-q^{n-k}})\binom{n-1}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}$$

As well as

$$\binom{n}{k}_q = (q^{n-k} + \frac{1-q^{n-k}}{1-q^k})\binom{n-1}{k-1}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}$$

■

Another way one might instead define a q-analogue to the q-binomial coefficients is by deforming the binomial theorem in some way. In particular, the binomial coefficients are given as the coefficients for polynomials of the form $(1+x)^n$, and well we don't have something perfectly analogous for the q-binomial coefficients we do get something pretty close.

Theorem 2.5. *We have the following*

$$\prod_{k=0}^{n-1}(1+q^k x) = \sum_{k=0}^{n} q^{\binom{k}{2}}\binom{n}{k}_q x^k$$

Proof. Write $P(x) = \prod_{k=0}^{n-1}(1+q^k x)$, then $P(x) = \sum_{k=0}^{n} a_k x^k$ for some coefficients a_k, clearly $a_1 = 1$. Note that

$$(1+x)P(qx) = (1+x)\prod_{k=1}^{n}(1+q^k x) = \prod_{k=0}^{n}(1+q^k x) = (1+q^n x)P(x)$$

So because $P(x) = \sum_{k=0}^{n} a_k x^k$ it follows that

$$a_k q^k + a_{k-1}q^{k-1} = a_k + q^n a_{k-1}$$

equivalently

$$a_k = \frac{q^n - q^{k-1}}{(q^k - 1)} a_{k-1}$$

Now note that $\binom{n}{1}_q q^{\binom{1}{2}} = 1$ and

$$\frac{q^n - q^{k-1}}{(q^k-1)}\binom{n}{k-1}_q q^{\binom{k-1}{2}} = \frac{q^{n-k+1}-1}{(q^k-1)} q^{k-1+\binom{k-1}{2}} \prod_{m=1}^{k-1}\frac{(1-q^{n-m+1})}{1-q^m} = q^{\binom{k}{2}}\prod_{m=1}^{k}\frac{(1-q^{n-m+1})}{1-q^m}$$

So the sequence $\binom{n}{k}_q q^{\binom{k}{2}}$ fulfills the same base case and recurrence as the coefficients a_k. Thus $a_k = \binom{n}{k}_q q^{\binom{k}{2}}$ and thus

$$\prod_{k=0}^{n-1}(1+q^k x) = \sum_{k=0}^{n} q^{\binom{k}{2}}\binom{n}{k}_q x^k$$

■

Corollary 2.6.

$$\prod_{k=0}^{n-1}\frac{1}{1-q^k x} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}_q x^k$$

Proof. Note that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\binom{n+k-1}{k}_q x^k + q^n x\sum_{k=0}^{\infty}\binom{n+k}{k}_q x^k \\
&= \sum_{k=0}^{\infty}\left(\binom{n+k-1}{k}_q + q^n\binom{n+k-1}{k-1}_q\right)x^k \\
&= \sum_{k=0}^{\infty}\left(\binom{n+k-1}{k}_q + q^{n+k-1-(k-1)}\binom{n+k-1}{k-1}_q\right)x^k \\
&= \sum_{k=0}^{\infty}\binom{n+k}{k}_q x^k
\end{aligned}$$

and so

$$\frac{1}{1-q^n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}_q x^k = \sum_{k=0}^{\infty}\binom{n+k}{k}_q x^k$$

. Thus $\sum_{k=0}^{\infty}\binom{n+k-1}{k}_q x^k$ fulfills the same recurrence as $\prod_{k=0}^{n-1}\frac{1}{1-q^k x}$ so they are equal. ■

You may have noticed that if we set $x = q$ and take $n \to \infty$ the functions

$$\prod_{k=0}^{n-1}\frac{1}{1-q^k x}$$

and

$$\prod_{k=0}^{n-1} 1 + q^k x$$

go to the generating functions for partitions and distinct partitions. So if the limits

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\binom{n+k-1}{k}_q q^k$$

and

$$\lim_{n\to\infty}\sum_{k=0}^{n} q^{\binom{k}{2}}\binom{n}{k}_q q^k$$

exist, then we should get interesting formulae for these generating functions. Luckily if $|q| < 1$ we have

$$\lim_{n\to\infty}\binom{n}{k}_q = \lim_{n\to\infty}\prod_{m=1}^{k}\frac{(1-q^{n-m+1})}{1-q^m} = \prod_{m=1}^{k}\frac{1}{1-q^m} = \frac{1}{[k]_q!(1-q)^k}$$

. Thus we have

$$\prod_{k=1}^{\infty}\frac{1}{1-q^k} = \sum_{k=0}^{\infty}\frac{q^k}{[k]_q!(1-q)^k}$$

and

$$\prod_{k=1}^{\infty} 1 + q^k = \sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}q^k}{[k]_q!(1-q)^k}$$

Or more generally

$$\prod_{k=0}^{\infty} \frac{1}{1-xq^k} = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!(1-q)^k} \tag{2.1}$$

and

$$\prod_{k=0}^{\infty} 1+xq^k = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}x^k}{[k]_q!(1-q)^k} \tag{2.2}$$

We shall now use these formulae to prove a very powerful theorem. The proof is lengthy but really just amounts to a few substitutions.

Theorem 2.7 (The Jacobi Triple Product)**.** *If* $|q| < 1$ *then*

$$\sum_{n=-\infty}^{\infty} q^{n^2}z^n = \prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+q^{2n-1}z^{-1})$$

Proof. From (2.2) we have

$$\prod_{k=0}^{\infty} 1+xq^k = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}x^k}{[k]_q!(1-q)^k} = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}x^k}{\prod_{j=1}^{k}(1-q^j)}$$

Now substitute $q \to q^2$ and $x \to qz$ to obtain

$$\prod_{n=1}^{\infty}(1+q^{2n-1}z) = \prod_{n=0}^{\infty}(1+q^{2n+1}z) = \sum_{k=0}^{\infty} \frac{q^{k^2}x^k}{\prod_{j=1}^{k}(1-q^{2j})} \tag{2.3}$$

Multiplying both sides by $\prod_{n=1}^{\infty} 1-q^{2n}$ as to eliminate the denominator of the right hand side, we obtain

$$\prod_{n=1}^{\infty}(1+q^{2n-1}z)(1-q^{2n}) = \sum_{k=-\infty}^{\infty}(q^{k^2}z^k\prod_{n=0}^{\infty}(1-q^{2(n+k+1)}) \tag{2.4}$$

Now substitute $q \to q^2$ and $x \to -q^{2(k+1)}$ in equation (2.2) to obtain

$$\prod_{n=0}^{\infty} 1-q^{2(n+k+1)} = \sum_{j=0}^{\infty} \frac{(-1)^j q^{j^2+2kj+j}}{\prod_{i=1}^{j}(1-q^{2i})}$$

Plug this into (2.4) to find

$$\prod_{n=1}^{\infty}(1+q^{2n-1}z)(1-q^{2n}) = \sum_{k=-\infty}^{\infty}(q^{k^2}z^k\sum_{j=0}^{\infty}\frac{(-1)^j q^{j^2+2kj+j}}{\prod_{i=1}^{j}(1-q^{2i})}) \tag{2.5}$$

$$= \sum_{k=-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{q^{k^2}z^k(-qz^{-1})^j}{\prod_{i=1}^{j}(1-q^{2i})} \tag{2.6}$$

$$= (\sum_{j=0}^{\infty}\frac{(-qz^{-1})^j}{\prod_{i=1}^{j}(1-q^{2i})})\sum_{k=-\infty}^{\infty}q^{k^2}z^k \tag{2.7}$$

The second to last equality comes from shifting the k index to $k-j$. Now from (2.1) we have

$$\prod_{k=0}^{\infty}\frac{1}{1-xq^k}=\sum_{k=0}^{\infty}\frac{x^k}{[k]_q!(1-q)^k}=\sum_{k=0}^{\infty}\frac{x^k}{\prod_{i=1}^{k}(1-q^i)}$$

Now substitute $q\to q^2$ and $x\to(-qz^{-1})$

$$\prod_{n=0}^{\infty}\frac{1}{1+q^{2n-1}z^{-1}}=\sum_{k=0}^{\infty}\frac{(-qz^{-1})^k}{\prod_{i=1}^{k}(1-q^{2i})}$$

Substitute this into (2.7) and find

$$\prod_{n=1}^{\infty}(1+q^{2n-1}z)(1-q^{2n})=\prod_{n=0}^{\infty}\frac{1}{1+q^{2n-1}z^{-1}}\sum_{k=-\infty}^{\infty}q^{k^2}z^k$$

multiply both sides by $\prod_{n=0}^{\infty}1+q^{2n-1}z^{-1}$ to finally obtain

$$\sum_{n=-\infty}^{\infty}q^{n^2}z^n=\prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+q^{2n-1}z^{-1})$$

■

The Jacobi triple product is extremely powerful, it can be used to discover many strange and astonishing facts.

Example. Let us try to find the number of purely integer coordinate points on the k sphere of radius r. The number of such points is then given by $|\{(x_1,x_2,\dots x_k)\in\mathbb{Z}^k:\sum x_i^2=r^2\}|$ which is the coefficient of q^{r^2} in the power series

$$(\sum_{n\in\mathbb{Z}}q^{n^2})^k$$

substituting $z\to 1$ in the jacobi triple product we find that this is the q^{r^2} coefficient of

$$\prod_{n=1}^{\infty}(1-q^{2n})^k(1+q^{2n-1})^{2k}$$

which is signifaicantly easier to calculate.

Corollary 2.8. *The pentagonal number theorem*

$$\prod_{n=1}^{\infty}(1-q^n)=\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{3n^2-n}{2}}=1+\sum_{n=1}^{\infty}(-1)^n(q^{\frac{3n^2-n}{2}}+q^{\frac{3n^2+n}{2}})$$

Proof. Substitute $q\to q^{3/2}$ and $z\to -q^{-1/2}$ in the jacobi triple product to obtain

$$1+\sum_{n=1}^{\infty}(-1)^n(q^{\frac{3n^2-n}{2}}+q^{\frac{3n^2+n}{2}})=\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{3n^2-n}{2}}=\prod_{n=1}^{\infty}(1-q^{3n})(1-q^{3n-2})(1-q^{3n-1})=\prod_{n=1}^{\infty}(1-q^n)$$

■

We can use this to obtain a recurrence for the integer partition function $p(n)$.

Corollary 2.9.

$$p(n) = \sum_{k=1}^{\infty}(-1)^{n+1}(p(n - \frac{3k^2 - k}{2}) + p(n - \frac{3k^2 + k}{2})$$

Proof. Note that $\prod_{n=1}^{\infty}(1-q^n)^{-1} = \sum_{n=0}^{\infty} p(n)q^n$ and so

$$1 = \left(\prod_{n=1}^{\infty}(1-q^n)\right)\left(\sum_{n=0}^{\infty} p(n)q^n\right) = \left(1 + \sum_{n=1}^{\infty}(-1)^n(q^{\frac{3n^2-n}{2}} + q^{\frac{3n^2+n}{2}})\right)\left(\sum_{n=0}^{\infty} p(n)q^n\right)$$

and so

$$\sum_{n=0}^{\infty} p(n)x^n = 1 + \sum_{n=0}^{\infty} p(n)q^n \sum_{n=1}^{\infty}(-1)^{n+1}q^{\frac{3n^2-n}{2}} + \sum_{n=0}^{\infty} p(n)q^n \sum_{n=1}^{\infty}(-1)^{n+1}q^{\frac{3n^2+n}{2}}$$

Expanding out the left hand side and equating coefficients we find

$$p(n) = \sum_{k=1}^{\infty}(-1)^{n+1}(p(n - \frac{3k^2 - k}{2}) + p(n - \frac{3k^2 + k}{2})$$

■

3. Combinatorial interpretations

3.1. **Permutation statistics.** So far we have discussed various ways one might go about defining q-analogs via various numerical descriptions, but many combinatorial objects are described via what they count. The factorial, for example, gives the number of permutations of the integers $1-n$. However we could refine this count with a generating function giving a q-analog.

Definition 3.1. Given a permutation σ, an inversion of σ is a pair $i < j$ such that $\sigma(i) > \sigma(j)$. The inversion count of σ is the total number of inversions of σ, so

$$\text{Inv}\sigma = \sum_{i<j, \sigma(i) > \sigma(j)} 1$$

. The inversion polynomial is then $\text{Inv}_n(q) = \sum_{\sigma \in S_n} q^{\text{Inv}\,\sigma}$.

The inversion polynomial seems like a perfectly valid q-analog to the factorial, after all $\text{Inv}_n(1) = \sum_{\sigma \in S_n} 1^{\text{Inv}\,\sigma} = \sum_{\sigma \in S_n} 1 = n!$. In fact it seems much more motivated then our current definition of the q-factorial, so why don't we use it. As it turns out it is exactly the same as the q-factorial.

Lemma 3.2. $\text{Inv}_n(q) = [n]_q!$

Proof. For any $\sigma \in S_{n-1}$ Let ${}^k\sigma \in S_n$ be the permutation obtained by inserting n between the $k-1$th and kth spots. So if $\sigma = 123$ then ${}^2\sigma = 1423$. Now note that because this construction leaves the relative order of all but the kth element the only new inversions added are $k < i$, ${}^k\sigma(i) < n$ for all $k < i$. Thus $\text{Inv}^k\,\sigma = \text{Inv}\,\sigma + n - k$. Clearly every element of S_n can be obtained as ${}^k\sigma$ for some k and some $\sigma \in S_{n-1}$ so

$$\text{Inv}_n(q) = \sum_{\sigma \in S_n} q^{\text{Inv}\,\sigma} = \sum_{\sigma \in S_{n-1}} \sum_{k=1}^{n} q^{\text{Inv}\,\sigma + n - k} = (1 + q^1 + \cdots + q^{n-1}) \sum_{\sigma \in S_{n-1}} q^{\text{Inv}\,\sigma} = [n]_q \,\text{Inv}_{n-1}(q)$$

Since $\mathrm{Inv}_1 = 1$, $[n]_q!$ and $\mathrm{Inv}_n(q)$ fulfill the same recurrence and base case and thus are equal. ∎

3.2. **Linear algebra over finite fields.** When q is a prime power there is a unique field $\mathbb{F}_q$ with exactly q elements. Linear algebra over these fields is fairly similar to linear algebra over any other field with one exception, we can count things. When trying to count things in these fields q-analogs become highly relevant. If we want to understand linear algebra over finite fields combinatorially a reasonable place to start would asking how many k-dimensional subspaces of $\mathbb{F}_q^n$ are there. The answer turns out to be very familiar

Theorem 3.3. *The number of k dimensional subspaces of $\mathbb{F}_q^n$ is $\binom{n}{k}_q$*

Proof. First note that the number of elements of a k dimensional subspace is exactly q^k. To find the number of k dimensional subspaces we shall first find the number of sets S of k linearly independent and then divide out by the number of different linearly independent sets that give the same subspace. To pick a linearly independent subset of size k we first pick any non-zero vector v_1, we have $q^n - 1$ options. Then we pick any vector v_2 not in the span of v_1, because there are q vectors in the span of v_1 we have $q^n - q$ options. For v_3 there are q^2 vectors in the span of v_1 and v_2 so $q^n - q^2$ options for v_3. So on and so forth until we get to v_k. We have imposed an order here so the total number of linearly independent sets of size k is $\frac{\prod_{m=0}^{k-1} q^n - q^m}{k!}$. To find the total number of linearly independent sets of size k that span a particular k dimensional subspace, note that this is the same as asking what the total number of linearly independent sets of size k is in $\mathbb{F}_q^k$. We have just found this for any n and k, setting $n = k$ we find that this is just $\frac{\prod_{m=0}^{k-1} q^k - q^m}{k!}$. Thus the total number of k dimensional subspaces of $\mathbb{F}_k^n$ is

$$\frac{\prod_{m=0}^{k-1} q^n - q^m}{\prod_{m=0}^{k-1} q^k - q^m} = \frac{\prod_{m=0}^{k-1}(q^{n-m} - 1)}{\prod_{m=0}^{k-1}(q^{k-m} - 1)} = \frac{\prod_{m=0}^{k-1}(1 - q^{n-m})}{\prod_{m=1}^{k}(1 - q^m)} = \binom{n}{k}_q$$

∎

Corollary 3.4. *The order of the general linear group $GL_n(\mathbb{F}_q)$ is given by $[n]_q!(q-1)^n q^{\binom{n}{2}}$.*

In the course of the above proof we showed that the number of length k ordered lists of linearly independent vectors in $\mathbb{F}_q^n$ is $\prod_{m=0}^{k-1}(q^k - q^m)$. Since an element of $GL_n(\mathbb{F}_q)$ is just a length n ordered list of linearly independent vectors in $\mathbb{F}_q^n$ it follows that

$$|GL_n(\mathbb{F}_q)| = \prod_{m=0}^{n-1}(q^n - q^m) = \prod_{m=0}^{n-1} q^m \times \prod_{m=0}^{n-1}(q^{n-m} - 1) = (q-1)^n q^{\binom{n}{2}} \prod_{m=1}^{n} \frac{(q^m - 1)}{q - 1} = [n]_q!(q-1)^n q^{\binom{n}{2}}$$

Interestingly the q factorial also counts a different, less well known, linear algebraic structure, called flags.

Definition 3.5. Let V be a vector space, a flag of length k in V is a sequence

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = V$$

. A complete flag is a flag of length $\dim V$.

Theorem 3.6. *The number of complete flags in $\mathbb{F}_q^n$ is equal to $[n]_k!$.*

Proof. To obtain a complete flag we first pick a vector v_1 which spans V_1, we have $q^n - 1$ options but every 1 dimensional subspace has $q-1$ vectors which span it, so we really have $\frac{q^n-1}{q-1}$. For V_2 we pick a vector v_2 not in V_1, the span of v_1, v_2 will be V_2, there are $q^n - q$ options but for any choice of v_2 there are $q^2 - q$ options that will give the same subspace, so we really only $\frac{q^n-q}{q^2-q}$. We continue on like this until we have picked n subspaces, so the total number of complete flags is

$$\prod_{k=0}^{n-1} \frac{q^n - q^k}{q^k(q-1)} = \prod_{k=0}^{n-1} \frac{q^{n-k} - 1}{q-1} = \prod_{k=1}^{n} \frac{q^k - 1}{q-1} = [n]_q!$$

■

Remark 3.7. (This remark is not necessary for the rest of the paper.) The counting formulas above are sometimes viewed as part more general phenomenon: many combinatorial structures behave as though they arise as the "$q \to 1$" specializations of linear algebra over a hypothetical object called the *field with one element*, denoted $\mathbb{F}_1$. The idea is that certain constructions over finite fields $\mathbb{F}_q$ have well-defined limits as $q \to 1$, and these limiting objects frequently turn out to be purely combinatorial. For example, the formulas for the number of subspaces of $\mathbb{F}_q^n$ and the number of complete flags both reduce to $n!$ when $q \to 1$, matching the interpretation of $n!$ as counting permutations.

This heuristic "$q \to 1$" principle is one motivation behind attempts to develop algebraic geometry over $\mathbb{F}_1$. We will not pursue this perspective further here, but it helps explain why so many q-analogues simultaneously generalize finite field phenomena and classical combinatorics.

4. q-calculus

Consider the limit

$$\lim_{q \to 1} \frac{f(qx) - f(x)}{qx - x}$$

Clearly this is the derivative. However if we instead fix q we get an interesting analog to the derivative.

Definition 4.1. For any function f, define

$$(D_q f)(x) = \frac{f(qx) - f(x)}{x(q-1)}$$

Example. Many functions have easily computable q-derivatives. For example

$$D_q \frac{1}{1-x} = \frac{\frac{1}{1-qx} - \frac{1}{1-x}}{qx - x} = \frac{(q-1)x}{x(q-1)(1-qx)(1-x)} = \frac{1}{(1-x)(1-qx)}$$

This fulfills many properties similar to the typical derivative

Lemma 4.2. *(i) D_q is linear*
(ii) (The q product rule) $D_q(fg)(x) = f(qx)D_q g(x) + (D_q f(x))g(x)$
(iii) (The q power rule) $D_q(x^n) = [n]_q x^{n-1}$
(iiii) (The q quotient rules) $D_q(\frac{f(x)}{g(x)}) = \frac{g(x)D_q f(x) - f(x) D_q g(x)}{g(qx)g(x)}$

Proof. (i) Let f, g be any functions and a, b be constants, then

$$D_q(af+bg)(x) = \frac{af(qx)+bg(qx)-af(x)-bg(x)}{qx-x} = aD_qf(x)+bD_qg(x)$$

(ii)

$$\begin{aligned} D_q(fg)(x) &= \frac{f(qx)g(qx)-f(x)g(x)}{x(q-1)} \\ &= \frac{f(qx)g(qx)-f(qx)g(x)+f(qx)g(x)-f(x)g(x)}{x(q-1)} \\ &= f(qx)D_qg(x)+(D_qf(x))g(x) \end{aligned}$$

(iii)

$$D_q(x^n) = \frac{q^nx^n-x^n}{x(q-1)} = x^{n-1}\frac{q^n-1}{(q-1)} = [n]_qx^{n-1}$$

(iiii) Differentiate both sides of $g(x)\frac{f(x)}{g(x)} = f(x)$ and apply the product rule to obtain

$$g(qx)D_q(\frac{f(x)}{g(x)}) + \frac{f(x)}{g(x)}D_qg(x) = D_qf(x)$$

Simplify to find

$$D_q(\frac{f(x)}{g(x)}) = \frac{g(x)D_qf(x)-f(x)D_qg(x)}{g(qx)g(x)}$$

■

Now I'm sure you're wondering where the chain rule is. Unfortunately, the sad truth is that there is no chain rule in q-calculus. Because of this, we can't be sure that $D_q(x-a)^n = [n]_q(x-a)^n$ even though it is true when $a=0$, in fact, this is almost never the case, generally we have

$$D_q(x-a)^n = \frac{(qx-a)^n-(x-a)^n}{x(q-1)}$$

which doesn't simplify to anything nice. Nonetheless we definitely still want some ability to "shift" x^n that is coherent with the q-derivative.

Definition 4.3. For any a define $(x-a)_q^n = \prod_{k=1}^n(x-q^{k-1}a)$ when $n \geq 1$ and $(x-a)_q^0 := 1$. (Often you will see the specific case $(1-a)_q^n$ notated $(a;q)_n$ and called the q-pochhammer symbol)

These do behave nicely with the q-derivative.

Proposition 4.4. $D_q(x-a)_q^n = [n]_q(x-a)_q^{n-1}$

Proof. We induct on n, when $n=1$ the claim is trivial so assume $n>1$ and the claim holds for all $k<n$. Then by the power rule we have

$$\begin{aligned} D_q(x-a)_q^n &= D_q(x-aq^{n-1})(x-a)_q^{n-1} \\ &= (x-a)_q^{n-1} + (qx-aq^{n-1})D_q(x-a)_q^{n-1} \\ &= (x-a)_q^{n-1} + q[n-1]_q(x-a)_q^{n-2}(x-aq^{n-2}) \\ &= (1+q[n-1]_q)(x-a)_q^{n-1} \\ &= [n]_q(x-a)_q^{n-1} \end{aligned}$$

So the claim holds by induction. ■

This tells us, for example that

$$D_q(x-a)(x-qa) = [2]_q(x-a)$$

This may not seem like that big of a deal, but because

- $(a-a)_q^n = 0$ for all n but 0.
- $\deg(x-a)_q^n = n$
- $D_q(x-a)_q^n = [n]_q(x-a)_q^{n-1}$

We have the ability to write any polynomial in a " q-derivative friendly way " this is essentially a q-taylor series.

Theorem 4.5. *Let $f(x)$ be a polynomial, then $\sum_{k=0}^{\infty}(D_q^n f)(a)\frac{(x-a)_q^n}{[n]_q!}$*

Proof. Note first that $D_q^k\frac{(x-a)_q^n}{[n]_q!} = \frac{(x-a)_q^{n-k}}{[k]_q!}$ when $k < n$ and is 0 otherwise. Now suppose f has degree k, let V_k be the $k+1$-dimensional vector space of polynomials of degree $\leq k$. Then the set $\{(x-a)_q^0, (x-a)_q^1, \ldots, (x-a)_q^k\}$ forms a basis for V_k (Each $(x-a)_q^j$ have different degree and there are $k+1$ of them). Now write

$$f(x) = \sum_{j=0}^{k} b_j \frac{(x-a)_q^j}{[j]_q!}$$

Then

$$D_q^n f(a) = \sum_{j=0}^{k} b_j D_q^n \frac{(a-a)_q^j}{[j]_q!} = \sum_{j=n}^{k} b_j \frac{(a-a)_q^{j-n}}{[j-n]_q!} = \sum_{j=0}^{k-n} b_{j+n} \frac{(a-a)_q^j}{[j]_q!} = b_n$$

So

$$\sum_{k=0}^{\infty}(D_q^n f)(a)\frac{(x-a)_q^n}{[n]_q!}$$

■

Example. For a simple example, let's examine the polynomial $x^2 + x$. Then, its first q-derivative is $[2]_q x + 1$ and its second q-derivative is just $[2]_q$. Thus for any a we have

$$x^2 + x = a^2 + a + ((1+q)a + 1)(x-a) + (x-a)(x-qa)$$

We can extend this to formal power series convergent about 0 in a fairly obvious way.

Corollary 4.6. *Let $f(x) = \sum_{k=0}^{\infty} f_k x^k$ be a formal power series convergent about 0, then $\sum_{k=0}^{\infty}(D_q^n f)(0)\frac{x^n}{[n]_q!}$*

Proof. Write $f(x) = \sum_{k=0}^{\infty}[k]_q! f_k \frac{x^k}{[k]_q!}$, then clearly

$$D_q^n f(0) = \sum_{k=0}^{\infty}[k]_q! f_k (D_q^n \frac{x^k}{[k]_q!})(0) = \sum_{k=0}^{\infty}[k]_q! f_k \frac{0^{k-n}}{[k-n]_q!} = [n]_q! f_n$$

So

$$f(x) = \sum_{k=0}^{\infty}(D_q^n f)(0)\frac{x^k}{[k]_q!}$$

■

Example. Suprisingly, this actually gives us quite a bit of knowledge about the q-derivatives of a particular function. For example, using the typical power series expansion we know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

But we also know now that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{D_q^n \frac{1}{1-x}(0)}{[n]_q!} x^n$$

So in particular we know that

$$D_q^n(\frac{1}{1-x})(0) = [n]_q!$$

Definition 4.7. We can now properly define the q-exponential. The issue is, that there are two different ways we could go about defining a q exponential, and for once they do not agree. The first way is to define

$$e_q^x := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$$

in which case $D_q e_q^x = e_q^x$. The second way is to deform the differential equation $\frac{d}{dx}\exp(x) = \exp(x)$ into $D_q \exp_q(x) = \exp_q(qx)$. In this case we get the function

$$\exp_q(x) := \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!}$$

Recall the formulae we found in section one

$$\prod_{k=0}^{n-1} \frac{1}{1-q^k x} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q x^k$$

and

$$\prod_{k=0}^{n-1} (1+q^k x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q x^k$$

Note that

$$\prod_{k=0}^{n-1} \frac{1}{1-xq^k} = \frac{1}{(1-x)_q^n}$$

and

$$\prod_{k=0}^{n-1} 1 + xq^k = (1+x)_q^n$$

Then these formulae amount to the statement that $\sum_{k=0}^{\infty} \binom{n+k-1}{k}_q x^k$ is the taylor expansion of $\frac{1}{(1-x)_q^n}$ and $\sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q x^k$ is the taylor expansion of $(1+x)_q^n$. In particular, because we can extend these formulae to $n \to \infty$ this means we can define

$$\frac{1}{(1-x)_q^{\infty}} = \prod_{k=0}^{\infty} \frac{1}{1-xq^k} = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!(1-q)^k} \tag{4.1}$$

and

$$(1+x)_q^{\infty} = \prod_{k=0}^{\infty} 1 + xq^k = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{[k]_q!(1-q)^k} \tag{4.2}$$

So we can express $\frac{1}{(1-x)_q^{\infty}}$ in terms of e_q^x

$$\frac{1}{(1-x)_q^{\infty}} = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!(1-q)^k} = \sum_{k=0}^{\infty} \frac{(\frac{x}{1-q})^k}{[k]_q!} = e_q^{\frac{x}{1-q}}$$

and $(1+x)_q^{\infty}$ in terms of $\exp_q$

$$(1+x)_q^{\infty} = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{[k]_q!(1-q)^k} = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} (\frac{x}{1-q})^k}{[k]_q!} = \exp_q(\frac{x}{1-q})$$

In addition we have

$$\exp_q(\frac{x}{1-q}) = (1+x)_q^{\infty} = \prod_{k=1}^{\infty}(1+q^k x) = \frac{1}{\prod_{k=1}^{\infty} \frac{1}{1+xq^k}} = \frac{1}{\frac{1}{(1-(-x))_q^{\infty}}} = \frac{1}{e_q^{\frac{-x}{1-q}}}$$

And so

$$e_q^{-x} \exp_q(x) = e_q^x \exp_q(-x) = 1$$

In particular this means that

$$\sum_{k=0}^{n} \frac{q^{\binom{k}{2}}(-1)^k}{[k]_q![n-k]_q!} = \begin{cases} 0 : n \geq 1 \\ 1 : n = 0 \end{cases}$$

5. Conclusion

The theory of q-analogues provides a unified framework in which classical combinatorial quantities, finite field enumeration, and deformations of algebraic formulas all appear as different manifestations of the same underlying structures. Beginning from q-integers and q-binomial coefficients, we saw how their algebraic properties lead naturally to product identities, generating functions, and ultimately the Jacobi Triple Product. Interpreting these objects combinatorially connects them to permutation statistics, while interpreting them over finite fields recovers the enumeration of subspaces, flags, and linear groups. Finally, q-calculus shows that many familiar analytic identities persist in deformed form and that the q-exponential functions are closely tied to the infinite products introduced earlier.

q-analogues sit at a crossroads: simple enough to admit explicit computation, flexible enough to encode deep phenomena, and rich enough to unify disparate areas. Even the elementary examples here hint at broader connections to partitions, basic hypergeometric series, representation theory, and the geometry of $\mathbb{F}_1$. The ubiquity of these constructions is ultimately what makes q-analogues so powerful: they reveal the hidden "q-structure" already present in classical mathematics.