The Borsuk-Ulam Theorem & Tucker's Lemma

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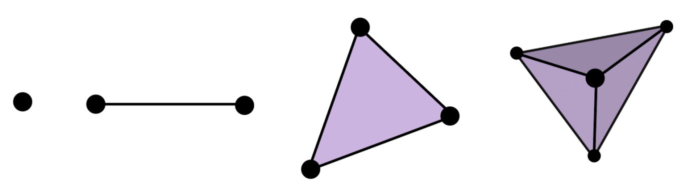
1 Introduction

Topology is related to combinatorics in some interesting ways. Our main goal in this paper is showing that the Borsuk-Ulam Theorem, a topological statement, is equivalent to Tucker's Lemma, a combinatorial statement. We'll then prove the Borsuk-Ulam Theorem through a proof of Tucker's Lemma. We'll end with a solution of the necklace splitting problem (complete with a story about pirates!) that uses the Borsuk-Ulam Theorem in its proof.

2 Background

First, we need to talk about simplices.

Definition 1. A **simplex** is the convex hull of a set A of linearly independent points in $\mathbb{R}^n$. The points in A are called the **vertices** of the simplex, denoted $V(A)$. Here are some examples of simplices:

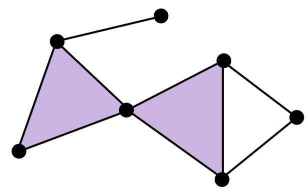


We can imagine the convex hull like this: Say that $|A| = n$. If we imagine an $(n-1)$-dimensional balloon containing all the points of A inside and deflate it, the resulting shape we get is a $(n-1)$-dimensional simplex. If we take the convex hull of a subset of points in A, we get a **face** of the simplex.

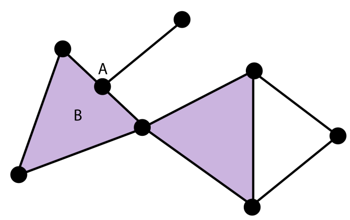
We can combine multiple simplices together to get a simplicial complex.

Definition 2. A **simplicial complex** is a collection of simplices Δ such that each face of a simplex $\sigma \in \Delta$ is also in Δ, and that for any two simplices $\sigma_1, \sigma_2 \in \Delta$, their intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

For example, this is a simplicial complex.



This is not a simplicial complex, because $A \cap B$ is not a face of B.

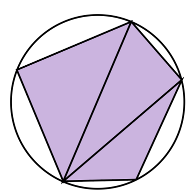


We call the union of all simplices in a simplicial complex Δ the **polyhedron** of Δ, denoted by $||\Delta||$.

Sometimes, we want to divide shapes into simplices. This is a called a triangulation.

Definition 3. A **triangulation** of a topological space X is a simplicial complex Δ such that $||\Delta|| \cong X$. ($X \cong Y$ means that there exists a homeomorphism $X \to Y$, so X can be smoothly deformed to become Y.)

For example, here is a triangulation of the ball B^2 :

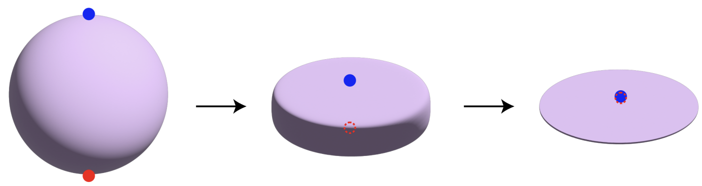


3 The Borsuk-Ulam Theorem and Tucker's Lemma

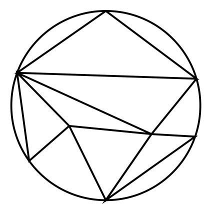
The Borsuk-Ulam Theorem is a topological statement, while Tucker's Lemma is a combinatorial statement. In this section, we'll prove that they're equivalent to each other.

Borsuk-Ulam Theorem [Bor33]. Let $f : S^n \to \mathbb{R}^n$ be a continuous mapping. Then, there exists a point $x \in S^n$ such that $f(x) = f(-x)$.

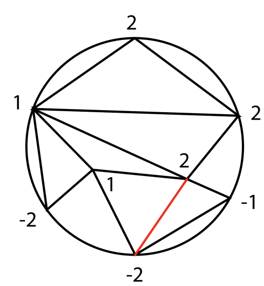
Essentially, what this theorem states is that if we take a ball living in $n+1$ dimensions and try to smoosh it in a smooth way down to a flat n-dimensional hyperplane, there will be two points that overlap with each other, that were originally on opposite sides of the surface of the ball. Here's an example of a possible smooshing when $n = 2$:



Tucker's Lemma is a bit more involved. Let T is a triangulation of B^n that's antipodally symmetric on the boundary of B^n (which is S^{n-1}). This means that if a point $x \in T$ is also on S^{n-1}, then we also know that $-x \in T$. Here's an example of a triangulation T when $n = 2$:



We label vertices of T with a number between $-n$ and n (but not 0) such that opposite points on S^{n-1} get opposite numbers. Formally, if $V(T)$ is the set of vertices of T, we define a function $\lambda : V(T) \to \{1, -1, 2, -2, \ldots, n, -n\}$ such that $\lambda(-x) = -\lambda(x)$ if $x \in S^{n-1}$.



Tucker's Lemma [Tuc45]. There will be a 1-dimensional simplex in T that has its two vertices labeled by opposite numbers. In the example above, it's colored in red.

3.1 Equivalence

Time to prove that these two statements are equivalent, through a bunch of side quests...

To start, we rewrite the two statements into more convenient forms.

Claim: Borsuk-Ulam Theorem $\iff$ There is no continuous mapping $f : B^n \to S^{n-1}$ such that for all $x \in S^{n-1}$, we have $-f(x) = f(-x)$. (1)

Proof: We first show the forward direction.

Assume the Borsuk-Ulam Theorem is true. Let $f : S^n \to \mathbb{R}^n$ be an antipodal mapping. (An antipodal mapping is one that for all x, we have $-f(x) = f(-x)$.) We know this mapping will have a point where $f(x) = f(-x) = 0$, by applying Borsuk-Ulam to f. Because of this, we can never have an antipodal mapping from $f : S^n \to S^{n-1}$ since any map that satisfies $-f(x) = f(-x)$ will have $f(x) = 0$ for some x.

Now, suppose we had a mapping $g : B^n \to S^{n-1}$ such that for all $x \in S^{n-1}$, we have $-g(x) = g(-x)$. Let π be the projection where $\pi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$. Then, consider the function $f : S^n \to S^{n-1}$ where

$$f(x) = \begin{cases} g(\pi(x)) & \text{if } x_n \geq 0 \\ -g(\pi(-x)) & \text{if } x_n < 0. \end{cases}$$

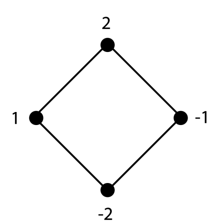
This function is continuous, because when $x_n = 0$, we have that $\pi(x) \in S^{n-1}$, which means that $-g(\pi(x)) = g(-\pi(x))$, so $g(\pi(x)) = -g(\pi(-x))$. This function is then an antipodal mapping from S^n to S^{n-1}, which is not possible, which must mean that there is no continuous mapping $f : B^n \to S^{n-1}$ such that for all $x \in S^{n-1}$, we have $-f(x) = f(-x)$. This is what we wanted to show!

The other direction goes through these functions in reverse. Suppose that there is no continuous mapping $g : B^n \to S^{n-1}$ such that for all $x \in S^{n-1}$, we have $-g(x) = g(-x)$. If we were to have an antipodal mapping $f : S^n \to S^{n-1}$ and we let $g(x) = f(\pi^{-1}(x))$ (we define the range of $\pi^{-1}(x)$ to be $S^n \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} \geq 0\}$), we would have a mapping where $-g(x) = g(-x)$ when $x \in S^{n-1}$. This means that the mapping f cannot exist.

Now, suppose we had an antipodal mapping $h : S^n \to \mathbb{R}^n$ where $h(x) \neq 0$. Then, the function $f(x) = \frac{h(x)}{||h(x)||}$ would satisfy the conditions above, so such an h cannot exist. This means an antipodal mapping $h : S^n \to \mathbb{R}^n$ must have some point where $h(x) = 0$.

Finally, if we have a continuous mapping $f : S^n \to \mathbb{R}^n$, we can create an antipodal mapping $h(x) = f(x) - f(-x)$, which means that $f(x) = f(-x)$ at some point. And this is the Borsuk-Ulam Theorem! Whew. As a fun little bonus, because of our proof, all these claims about these different kinds of functions we made along the way are all equivalent to the Borsuk-Ulam Theorem.

To reformulate Tucker's lemma, we first define a special simplicial complex $\diamond^{n-1}$. If we have a set of points $V(\diamond^{n-1}) = \{1, -1, 2, -2, \ldots, n, -n\}$, we can define a simplicial complex by saying that any subset $V' \subseteq V(\diamond^{n-1})$ is a simplex within the complex if V' does not simultaneously contain two opposite numbers. For example, here is $\diamond^1$ (note that the simplices are dots and lines).



Claim: Tucker's Lemma applies on triangulation $T \iff$ there does not exist a map $\lambda : V(T) \to V(\diamond^{n-1})$ that preserves simplices when going from T to $\diamond^{n-1}$ and is also antipodal on the boundary. (2)

Proof: We see that λ takes the vertices in T and maps them to vertices in $\diamond^{n-1}$, effectively assigning each vertex a number between $-n$ and n. Tucker's Lemma states that one of the simplices S in T will contain opposite numbers. However, if λ preserves simplices when going from T to $\diamond^{n-1}$, the simplex that S corresponds to in $\diamond^{n-1}$ cannot exist by how we defined simplices in $\diamond^{n-1}$. Thus, there cannot exist such a λ.

Now we're ready to show equivalence!

Claim: (1) $\implies$ (2).

We prove the contrapositive. Suppose that λ in (2) actually exists. Then, we can use the affine extension of λ to construct a function that "fills in the gaps" of the domain of λ to become B^n. (See page 15 of [Mat03] for more details on how this function is defined.) Let this function be $||\lambda||$. Then, since $\diamond^{n-1} \cong S^{n-1}$, we can use $||\lambda||$ to create a continuous map from B^n to S^{n-1} that is antipodal on the boundary. This contradicts (1).

Claim: (2) $\implies$ (1).

Again, we prove the contrapositive. Assume we have a continuous map $f : B^n \to S^{n-1}$ that's antipodal on S^{n-1}. We'll use f to construct a triangulation T and λ that contradicts (2).

Let's start by constructing T. The only constraint we need for T is for all simplices in T to have a diameter (defined as the greatest distance between any two vertices) less than δ, for some δ we choose.

Let's define what δ is. Let $\epsilon = \frac{1}{\sqrt{n}}$, meaning that for every $y \in S^{n-1}$, we have that some component y_i of y satisfies $y_i \geq \epsilon$. This is because that since $y \in S^{n-1}$, we have $\sqrt{y_1^2 + y_2^2 + \cdots + y_n^2} = 1$.

Now, because f is a continuous function on B^n, which is a compact set, it must be uniformly continuous. This means that we can choose δ such that if $x, x' \in B^n$ are less than δ apart, we have that no component of $f(x) - f(x')$ is greater than 2ϵ. This is how we define δ for T: it's a surprise tool that will help us later.

We can now construct our labeling function λ. Let $k(x) = \min\{i : |f(x)_i| \geq \epsilon\}$, where $f(x)_i$ denotes the ith component of $f(x)$. We're guaranteed to have $\{i : |f(x)_i| \geq \epsilon\}$ be nonempty, by our definition of ϵ. Now, we set our labeling λ to be

$$\lambda(x) = \begin{cases} k(x) & \text{if } f(x)_{k(x)} > 0 \\ -k(x) & \text{if } f(x)_{k(x)} < 0. \end{cases}$$

This labeling satisfies $\lambda(-x) = -\lambda(x)$ when $x \in S^{n-1}$, since f is antipodal on S^{n-1}. Tucker's Lemma guarantees that there is a simplex in T consisting of points $\{v, v'\}$ where $\lambda(v) = -\lambda(v') > 0$. Then, by how we defined our functions, $f(v)_{\lambda(v)} \geq \epsilon$ and$f(v')_{\lambda(v)} \leq -\epsilon$. This means that $(f(v) - f(v'))_{\lambda(v)} \geq 2\epsilon$, which gives us a contradiction, based on how we chose δ. Thus, our claim is true.

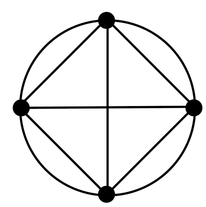
This means that we've now shown that the Borsuk-Ulam Theorem is equivalent to Tucker's Lemma. Hooray!

3.2 Proving Tucker's Lemma

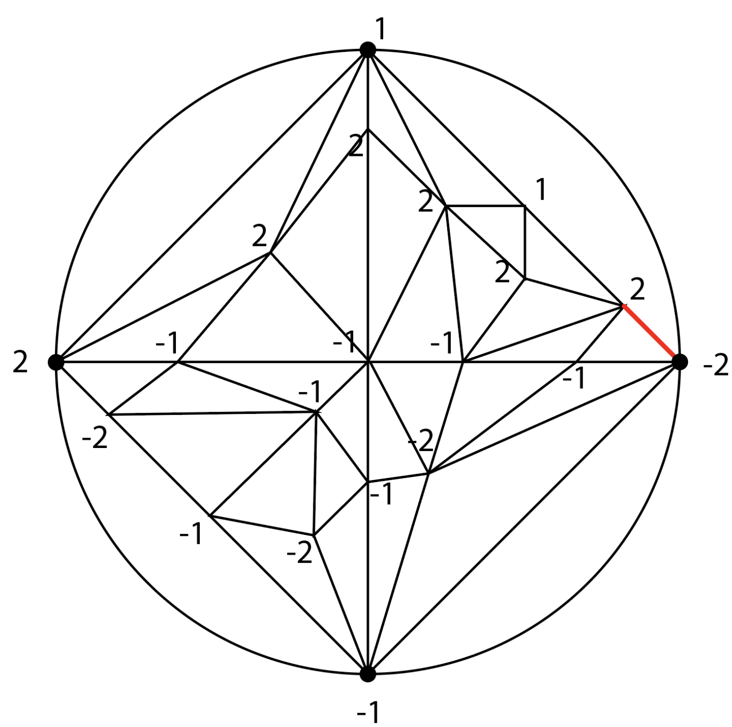
In our proof that Tucker's Lemma $\implies$ (1), our only requirement for T was that all simplices in T have a diameter less than some δ. In this proof, which follows [FT81], we'll assume some more things about T to make the proof easier, proving a weaker version of Tucker's Lemma.

For a proof for the general Tucker's Lemma, after finishing this proof, we can follow the above logic for Weaker Tucker's Lemma $\implies$ (1) $\implies$ (2) $\implies$ Tucker's Lemma.

Okay, here we go! Imagine we have a ball B^n centered at the origin. The set of n hyperplanes formed by setting one of the n coordinates to 0 triangulate the ball into 2^n sections. We call this a hyperplane triangulation. For example, here is a hyperplane triangulation of the ball B^2 :



We can refine hyperplane triangulations by triangulating the simplices of the hyperplane triangulation (and making sure the triangulation is antipodally symmetric along the boundary). A possible refinement looks like this, with labels added for Tucker's Lemma (the triangles that are formed are 2-dimensional simplices–I didn't shade them to make the diagram less chaotic). Note that the red 1-dimensional simplex contains opposite numbers.



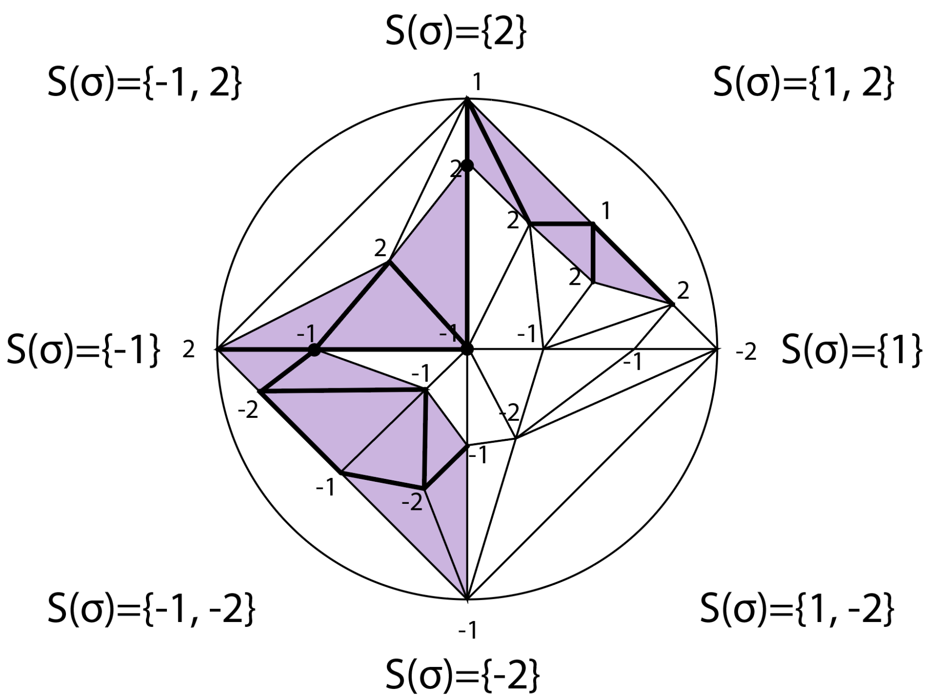
We'll prove Tucker's Lemma on these special triangulations. The simplices in these triangulations can be as small as we want, so our proof above can hold. Our proof is a proof by contradiction, so we're assuming that there does not exist a simplex with opposite numbers in T.

For each simplex σ, let $\lambda(\sigma)$ be the set of labels attached to the vertices of σ. For example, the red simplex has $\lambda(\sigma) = \{-2, 2\}$. Let's also define a function S on σ where

$$S(\sigma) = \{i : x_i > 0\} \cup \{-i : x_i < 0\},$$

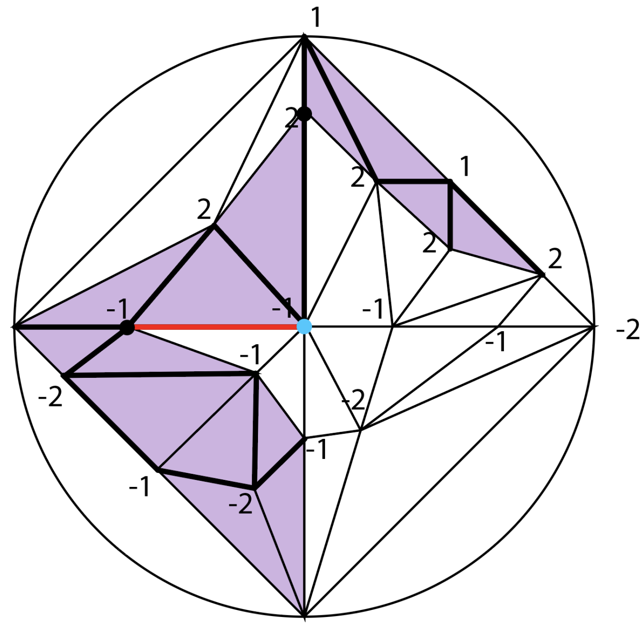
for some x in the interior of σ. The function $S(\sigma)$ is well-defined, since we know that all points in the interior of σ are on the same side of the coordinate axes, because of our restriction on T. As an example, the red simplex has $S(\sigma) = \{1, 2\}$.

Now, let's define a **good** simplex σ as one where $S(\sigma) \subseteq \lambda(\sigma)$. The red simplex is not good, since $\{1, 2\}$ is not a subset of $\{-2, 2\}$. Here are all the good simplices in our example triangulation (with labels for $S(\sigma)$):



Let's define a graph G whose vertices are the good simplices of T, where simplices $\sigma, \sigma' \in T$ are connected if they are either antipodal simplices on the boundary, or if σ is a face of σ' where $\lambda(\sigma) = S(\sigma')$. We'll prove that, if Tucker's Lemma is false, this graph will have exactly one vertex with an odd degree, which gives us a contradiction.

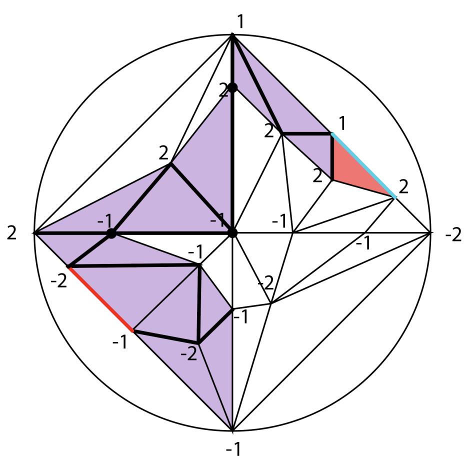
First of all, let's find our vertex with an odd degree. The zero-dimensional simplex O at the origin is connected to exactly one other good simplex: in this case, since $\lambda(O) = -1$, the highlighted two simplices are connected.



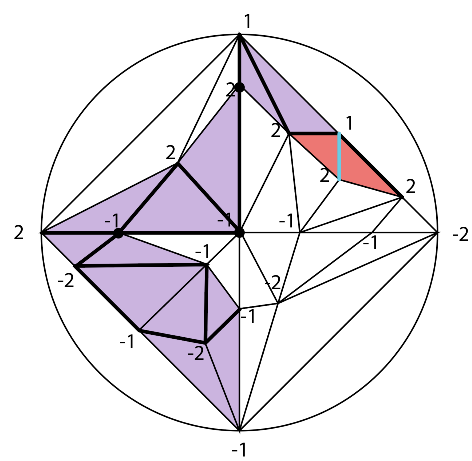
Now we need to show that all other good simplices are connected to two simplices.

Let $k = |S(\sigma)|$. Good simplices must have at least dimension $k-1$, since they must contain at least k points to have $S(\sigma) \subseteq \lambda(\sigma)$. We also know that σ must be in the space spanned by the k vectors corresponding to the coordinate axes, so the maximum dimension σ can have is k. This means that all good simplices have dimension k or $k-1$.

Case 1: A good simplex has dimension $k-1$. This simplex could either lie on the boundary or not. If it lies on the boundary, it's connected to its antipodal simplex and a neighboring simplex that repeats one of its labels (because otherwise we would have a simplex with opposite numbers). If not, it's connected to two simplices that must repeat its labels, by similar logic. In the diagram below, the blue simplices have dimension $k-1$, and are each connected to the two red simplices.

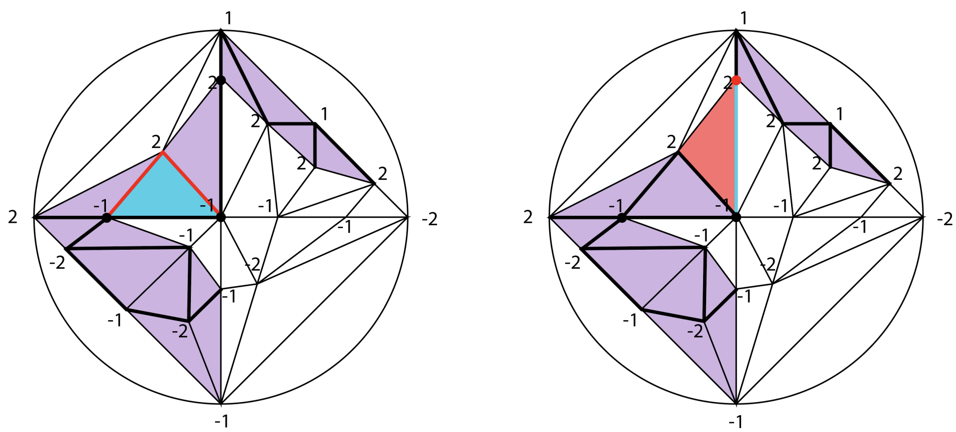


σ is on boundary

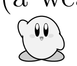


σ is not on boundary

Case 2: A good simplex has dimension k. This simplex either has a repeating label on one of its vertices, or has some label not in $S(\sigma)$. If it has a repeating label, it must have two faces that do not have repeating labels, which are both good simplices. If there is an extra label, then one of σ's connections must be the face without the extra label. The other connection is the good simplex that has σ as a facet.



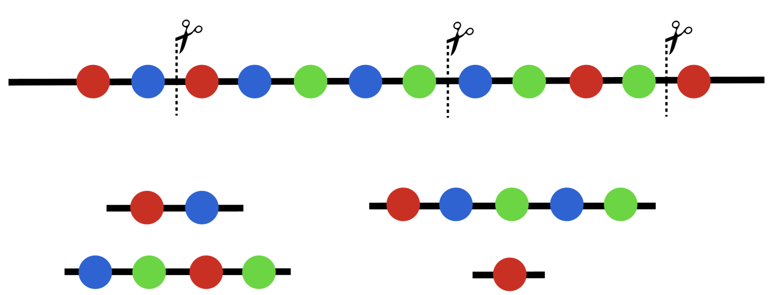
σ has repeating label σ has no repeating labels

In every case, each vertex has degree 2. This means that our graph has exactly one vertex with an odd degree, which gives us a contradiction. We've proved (a weak version) of Tucker's Lemma, which means we've proven the Borsuk-Ulam Theorem! 

4 The Necklace Splitting Problem

As a bonus, we'll end this paper off with the necklace splitting problem, which has flavortext that gives off Ted-Ed riddle vibes.

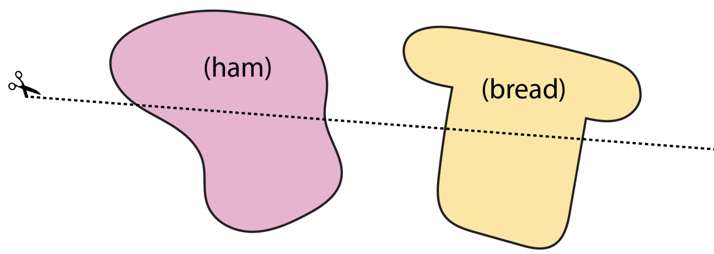
Here's the lore: you and your fellow pirate friend have found a necklace made of a bunch of different (suspiciously sphere-shaped) jewels! You want to divide up the jewels fairly, so that you both get the same number of jewels each. What's the minimum number of cuts you need to make to the necklace so that you each get an equal number of jewels? Here's an example, with three kinds of jewels:



Our claim is that for d jewels, the minimum number of cuts required is no more than d. To prove this, we'll first prove the ham sandwich theorem.

4.1 The Ham Sandwich Theorem

Intuitively, the ham sandwich theorem states that given a piece of ham, cheese, and bread floating in space, we can make a single cut that bisects the ham, cheese, and bread. More generally, d masses living in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane. Behold, my beautiful illustration where $d = 2$:



To state this formally, let's first define a finite Borel measure, which is a measure μ on $\mathbb{R}^d$ such that all open sets are measurable and $0 < \mu(\mathbb{R}^d) < \infty$. Intuitively, all the mass is concentrated in a finite space, so that $\mu(\mathbb{R}^d)$ is finite.

Ham Sandwich Theorem. If $\mu_1, \mu_2, \ldots, \mu_d$ are finite Borel measures on $\mathbb{R}^d$ where a hyperplane has measure 0 for all μ_i, then there exists a hyperplane that splits $\mathbb{R}^d$ into two halfspaces h^+ and h^- such that

$$\mu_i(h^+) = \mu_i(h^-) = \frac{1}{2}\mu_i(\mathbb{R}^d)$$

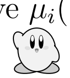
for all i.

To prove this, we're ultimately going to define a function $f : S^d \to \mathbb{R}^d$ that we can use the Borsuk-Ulam theorem on. First, let's associate each point $u = (u_0, u_1, \ldots u_d) \in S^d$ to a halfspace by defining

$$h^+(u) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : u_1x_1 + \cdots + u_dx_d \leq u_0\}.$$

Antipodal points of S^d correspond to opposite half-spaces, because $h^+(u)$ corresponds to $u_1x_1+\cdots+u_dx_d \leq u_0$ while $h^+(-u)$ means $-u_1x_1-\cdots-u_dx_d \leq -u_0$, or $u_1x_1+\cdots+u_dx_d \geq u_0$. Our function $f : S^d \to \mathbb{R}^d$ is then defined by

$$f(u) = (\mu_1(h^+(u)), \mu_2(h^+(u)), \ldots, \mu_d(h^+(u))).$$

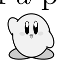
This function is continuous (though we're skipping the proof), which means we can apply the Borsuk-Ulam theorem to get that there exists a $x \in S^d$ such that $f(x) = f(-x)$. This means that for all i, we have $\mu_i(h^+(x)) = \mu_i(h^+(-x))$, so the hyperplane that bounds $h^+(x)$ is the hyperplane we want. 

From this version of the ham sandwich theorem, we can do a bit of work (see Chapter 3.1 of [Mat03]) to get the discrete form of the ham sandwich theorem, which gives a hyperplane that bisects sets of points $A_1, A_2, \ldots, A_d \subset \mathbb{R}^d$.

4.2 Back to Necklaces

To finish off our necklace proof, we string the necklace along the moment curve in $\mathbb{R}^d$. As a reminder, the moment curve C is the curve in $\mathbb{R}^d$ where

$$C = \{c : c = (x, x^2, x^3, \ldots, x^d), x \in \mathbb{R}\}.$$

Define A_i as the set of locations of the jewels of type i along the moment curve. We can use the discrete ham sandwich theorem to find a hyperplane that bisects all A_i. Since one of the properties of the moment curve is that a hyperplane can only intersect the moment curve in a maximum of d places, this then means that we need at most d cuts to evenly split our necklace. Hooray! 

5 Bibliography

[Bor33] Karol Borsuk. Drei sätze über die n-dimensionale euklidische sphäre. *Fundamenta Mathematicae*, 20(1):177–190, 1933.

[FT81] Robert M Freund and Michael J Todd. A constructive proof of tucker's combinatorial lemma. *Journal of Combinatorial Theory, Series A*, 30(3):321–325, 1981.

[Mat03] J. Matousek. *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*. Universitext (Berlin. Print). Springer, 2003.

[Tuc45] A.W. Tucker. *Some Topological Properties of Disk and Sphere*. 1945.