TOPOLOGICAL COMBINATORICS

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ABSTRACT. Topological combinatorics is the application of topological methods to solve combinatorial problems. This paper will discuss the applications of the Borsuk–Ulam theorem and combinatorial homotopy theory. We emphasize the relationship between the continuous nature of certain topological results and their discrete corollaries.

1. Introduction

Topological combinatorics was conceived in 1978 with Lászlo Lovász's seminal proof of the Kneser conjecture using the Borsuk–Ulam theorem, which remains central to the field; later, Tucker's lemma was proven its direct analogue in the discrete domain. Whereas the earlier discipline of combinatorial topology studied the arisal of topological structures from combinatorial ones (later, this became algebraic topology), the application of topology to discrete problems was new. More modern problems of topological combinatorics involve the use of Borsuk–Ulam and homotopy theory to solve partitioning and fair division problems: necklace-splitting and the ham sandwich theorem.

2. Topological Methods

We begin by laying the necessary topological groundwork.

Definition 2.1. A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \in X$ is a system of sets that satisfies the following properties:

(1) Both $\varnothing$ and X are in $\mathcal{T}$.
(2) The intersection of finitely many sets in $\mathcal{T}$ is in $\mathcal{T}$.
(3) The union of any arbitrary collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets* of X, and $\mathcal{T}$ is a *topology* on X.

Definition 2.2. For a topological space $(X, \mathcal{T})$, every subset $Y \subseteq X$ defines a topological space $(Y, \{U \cap Y : U \in \mathcal{T}\})$. Such a Y is called a *subspace*.

On topological spaces, we define certain key concepts.

Definition 2.3. Two functions $f, g : X \to Y$ on topological spaces X and Y are considered *homotopic*, denoted $f \sim g$, if there exists a family of maps $f_{t\in[0,1]}$, $f_0 = f$ and $f_1 = g$, such that there is a *homotopy* $H(x,t) = f_t(x)$ between $x \times [0,1]$ and Y that is continuous.

We can visualize the idea of homotopy by considering the two functions as different paths between two points in space. They are homotopic if one may be deformed continuously into the other. Moreover, based on homotopy, we can define an equivalence relation between spaces.

Definition 2.4. Two topological spaces X and Y are *homotopy equivalent* (i.e. belong to the same *homotopy class*) if there exist continuous maps $f, g : X \to Y$ such that $f \circ g \sim \mathrm{id}(X)$ and $g \circ f \sim \mathrm{id}(Y)$.

A stronger form of homotopy equivalence is useful in illustrating how certain visual representations of topological spaces we will encounter are identical.

Definition 2.5. A *homeomorphism* of topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ is a bijection $\varphi : X_1 \to X_2$ such that φ and φ^{-1} are both continuous. If φ exists, we write $X_1 \cong X_2$ (X_1 is homeomorphic to X_2).

Example. We can visualize a homeomorphism as a continuous deformation of one shape into another. Consider the longstanding quip that a torus is topologically equivalent to a coffee mug.

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Definition 2.6. The hypersphere S^d is the locus of all points in R^{d+1} equidistant from a common center.

We can also define a homotopy group on functions.

Definition 2.7. The nth homotopy group of X is the set of maps from S^n with some given base point s_0 into X with some given base point x_0 such that s_0 is mapped to x_0.

Definition 2.8. A topological space X is n-connected if its first n homotopy groups are trivial (contain only the identity function). The largest n such that this is true is denoted $\text{conn}(X)$.

Example. S^1 and S^2 are the 1-dimensional circumference of the circle and the 2-dimensional surface of the sphere, respectively.

Now, we consider simplicial complexes.

Definition 2.9. A *simplicial complex* $\mathcal{K}$ is a finite collection of nonempty finite sets $X \in \mathcal{K}$, such that for any X, every subset $Y \in X$ is also an element of $\mathcal{K}$. The union of all members of $\mathcal{K}$ is denoted $V(\mathcal{K})$; that is $\mathcal{K}$ can be said to be a set of subsets of $V(\mathcal{K})$, the elements of which are called the *vertices* of $\mathcal{K}$. The elements of $\mathcal{K}$ are called its *simplices*.

Definition 2.10. Given a simplicial complex $\mathcal{K}$, we construct the *geometric realization* $\tilde{\mathcal{K}}$ of $\mathcal{K}$. Embed $V(\mathcal{K})$ in $(|V(\mathcal{K})| - 1)$-dimensional space in general position (i.e. so that the points representing the vertices are not contained in one hyperplane). Then $\tilde{\mathcal{K}}$ is the union of all sets $\text{conv}(S)$, $S \in \mathcal{K}$. These convex hulls are called the *faces* of $\tilde{\mathcal{K}}$. We have for $S_1, S_2 \in \mathcal{K}$,

$$\text{conv}(S_1 \cap S_2) = \text{conv}(S_1) \cap \text{conv}(S_2).$$

Example. If $\mathcal{K}$ is the complete complex on 3 vertices, $\tilde{\mathcal{K}}$ is the union of the vertices, open edges, and interior of a triangle.

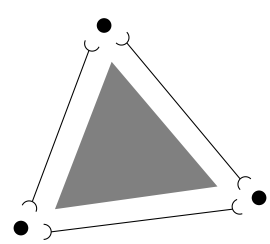


FIGURE 1. Simplicial complex $\mathcal{K}_3$.

The 1-dimensional simplex is the line, the 2-dimensional simplex is the triangle, the 3-simplex is the tetrahedron, and so on.

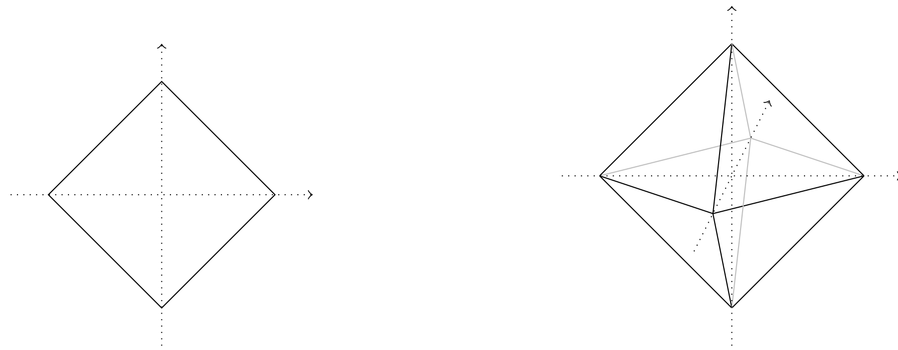
Definition 2.11. The *dimension* of a simplex $S \in \mathcal{K}$ is $\dim(S) = |S| - 1$; it is the number of dimensions of space in which it is imbedded.

Definition 2.12. The *d-dimensional crosspolytope* is the convex hull of the vectors in d-dimensional space of the standard orthonormal basis and their negatives.

Definition 2.13. Given a topological space X, a *triangulation* of X is a simplicial complex $\mathcal{K}$ such that $X \cong \tilde{\mathcal{K}}$.

Example. The simplest triangulation of the sphere S^{n-1} is the boundary of an n-simplex; for $n = 2$, that is the edge of the triangle; for $n = 3$, that is the surface of the tetrahedron.

Example. Below is an arbitrary triangulation of the disk (the union of the interior and boundary of the circle).



Two-dimensional crosspolytope. Three-dimensional crosspolytope.

FIGURE 2. Crosspolytope examples.

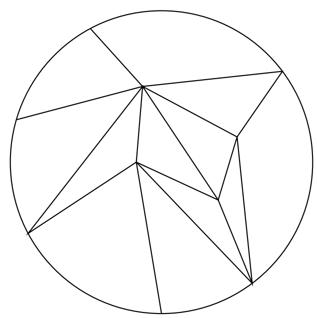


FIGURE 3. Triangulation of B^2.

3. Borsuk–Ulam Theorem and Analogues

The study of topological combinatorics depends largely on the combinatorial applications of the Borsuk–Ulam theorem, which has many equivalent statements. Here, we provide six.

Theorem 3.1 (Borsuk–Ulam)**.** *The following statements are equivalent, and true.*

(i) *For every continuous mapping* $: S^d \to \mathbb{R}^d$*, there exists* $x \in S_n$ *such that* $f(x) = f(-x)$*.*
(ii) *An antipodal mapping* $f : S^d \to \mathbb{R}^d$ *is a continuous mapping* f *such that* $f(-x) = -f(x)$ *for all* $x \in S^d$*. For every such* f *there exists a point* $x \in S^d$ *such that* $f(x) = 0$*.*
(iii) *For any cover* $F_1, \dots, F_{d+1}$ *of the sphere* S^d *by* $d+1$ *by* $d+1$ *closed sets, at least one set contains a pair of antipodal points.*
(iv) *For any cover* $U_1, \dots, U_{d+1}$ *of the sphere* S^d *by* $d+1$ *by* $d+1$ *open sets, at least one set contains a pair of antipodal points.*
(v) *There is no continuous mapping* $f : B^d \to S^{d-1}$ *that is antipodal on the boundary* S^{d-1}*.*
(vi) *If there exists a continuous antipodal mapping* $f : S^n \to S^m$*, then* $n \leq m$*.*

Example. For $d = 2$, Borsak–Ulam (i), informally, states that if a "balloon animal" made from the balloon S^2 is continuously flattened to $\mathbb{R}^2$, there exists a pair of points antipodal on the original balloon that will end up on top of each other. One can also visualize Borsuk–Ulam (iii) and (iv) for S^2.

The first result of the combinatorial application is that this theorem has a remarkable discrete analogue. Recall how a triangulation of the ball B^n partitions it into (sections homeomorphic to) the n-dimensional simplices; these are discrete. Tucker's lemma is a statement on these triangulations that is analogous to Borsuk–Ulam (v) on the original ball.

Lemma 3.2 (Tucker)**.** *Let* $\mathcal{T}$ *be a triangulation of* B^d *that is antipodally symmetric on the boundary. Label the vertices*

$$\lambda : V(\mathcal{T}) \to \{+1, -1, +2, -2, \dots, +n, -n\}$$

such that λ is antipodal on the boundary. Then there exists a complementary edge, i.e. a 1*-simplex whose vertex sum is* 0*.*

Example. In this labeling of the aforementioned triangulation of the disk, the complementary edge is outlined in red.

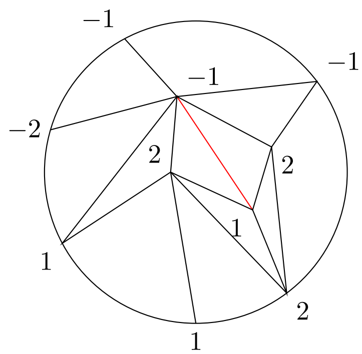


FIGURE 4. Labeled triangulation of B^2.

Proof. First, we restate the lemma. Let $\mathcal{L}^{n-1}$ denote the simplicial complex on

$$V(\mathcal{L}^{n-1}) = \{+1, -1, +2, -2, \dots, +n, -n\}$$

such that $F \subseteq V$ form a simplex if there exists no $i \in [n]$ such that $i \in F$ and $-i \in F$. This is simply the boundary of the d-dimensional crosspolytope, so it is homeomorphic to S^{n-1}. Thus the condition of Tucker's lemma is equivalent to the condition that there exists no simplicial map λ of $\mathcal{T}$ into $\mathcal{L}^{n-1}$ that is antipodal on the boundary.

We then wish to prove that this is equivalent to Borsuk–Ulam (v), whose condition is also on the non-antipodality of a map on the boundary.

Given Borsuk–Ulam (v), assume for the sake of contradiction the negation of Tucker. Then if there existed such a map λ, then its canonical affine extension would be a continuous map $B^n \to S^{n-1}$ antipodal on the boundary.

Conversely, assume Tucker; we prove the contrapositive of its implication of Borsuk–Ulam. Assume that there exists $f : B^n \to S^{n-1}$ antipodal on the boundary. Then we construct a triangulation $\mathcal{T}$ and mapping λ as follows.

Consider the coordinates of points $y \in S^{n-1}$. Let $\varepsilon = \frac{1}{\sqrt{n}}$; y must have at least one coordinate $y_i \geq \varepsilon$; otherwise

$$1 = ||y|| = \sum y_i^2 < n\varepsilon^2 = 1.$$

Then by the ε–δ definition of continuity there exists $\delta > 0$ such that if $||x - x'|| < \delta$, then $||f(x) - f(x')|| \leq 2\varepsilon$. We then choose $\mathcal{T}$ such that it has simplicial diameter at most δ.

Now, we define $\lambda : V(\mathcal{T}) \to \{+1, -1, \dots, +n, -n\}$. First let $k(v)$ be the value of i that yields smallest $|f(v)_i|$ that is still at least ε. Then

$$\lambda(v) = \begin{cases} +k(v) & \text{if } f(v)_{k(v)} > 0, \\ -k(v) & \text{if } f(v)_{k(v)} < 0. \end{cases}$$

Since f is antipodal on the boundary of B^n, we have λ is antipodal on the boundary as well. Thus by Tucker's lemma there exists some complementary edge vv' such that $\lambda(v) = -\lambda(v') = i$ for some i. Then $f(v)_i \geq \varepsilon$ and $f(v')_i \leq \varepsilon$, so $||f(v) - f(v')|| \geq f(v)_i - f(v')_i geq 2\varepsilon$; we have a contradiction. □

4. KNESER'S CONJECTURE

The first instance of a combinatorial problem solved using Borsuk–Ulam is Lovász's proof of the Kneser conjecture in [Lov78]. First, we present a different proof authored by Bárány [Bár78].

Definition 4.1. For vector $a \in S_d$, $H(a)$ denotes the set of points in S_d that lie on the same side of h as a, where h is the hyperplane orthogonal to a passing through the center of the sphere.

Theorem 4.2 (Gale)**.** *For nonnegative integers n and k, there exists a set $V \subset S_k$ with $2n+k$ elements such that $|H(a) \cap V| \geq n$ for each $a \in S_k$.*

Theorem 4.3 (Kneser–Lovász)**.** *If the n-subsets of $[2n+k]$ subset are partitioned into $k+1$ classes, at least one class will contain two disjoint n-subsets.*

Proof. Construct a set V, $|V| = 2n+k$, as in Gale's theorem. For the sake of contradiction assume there exists a $(k+1)$-coloring of the n-subsets of V. This yields a coloring of S_k where a point x has the color of every n-subset in $H(a) \cap V$; by Gale's theorem at least one exists.

Consider open sets of points of the same color; their union is S_k By version (iii) of Borsuk–Ulam on these sets, there exists a pair of antipodal points $x_0, -x_0$ of the same color. The n-tuples from which the two points acquired this coloring are disjoint because they lie on the disjoint hemispheres $H(x_0)$ and $H(-x_0)$, and they are of the same color. □

Lovász's own proof starts from a graph-theoretic formulation of Kneser's conjecture.

Definition 4.4. The *chromatic number* $\chi(G)$ is the minimum number of colors needed to color the vertices of G such that no two adjacent vertices are the same color.

Definition 4.5. A *Kneser graph* $K(n,k)$ is the graph whose vertices are the k-element subsets of a set of n elements where two vertices are connected by an edge iff their subsets are disjoint.

Theorem 4.6 (Kneser–Lovász, graph-theoretic)**.** *The chromatic number of $K(2n+k,n)$ is $k+2$. With an alternate definition of variables, this is*

$$\chi(K(n,k)) = n - 2k + 2.$$

Proof. First, note that we can easily construct a coloring of $K(2n+k,n)$ with $k+2$ colors. Partition the n-subsets of $[2n+k]$ into sets K_i, $1 \leq i \leq n+k+1$, such that a subset whose smallest element is i is placed in class K_i. Then consider the classes

$$K_1, K_2, \cdots, K_{k+1} \text{ and } K_0 = K_{k+2} \cup K_{k+3} \cup \cdots \cup K_{n+k+1}.$$

Any two elements in class K_i, $1 \leq i \leq k+1$, share element i. Any two elements in K_0 cannot be disjoint because they are n-sets chosen from among $|k+2, k+3, \ldots, 2n+k| = 2n-1$ elements.

Now, let's work on the graph $K(n,k)$ instead of $K(2n+k,n)$, for convenience; any following references to K, n, and k refer here. It remains to show that $K(n,k)$ is not $(n-2k+1)$-colorable. First, we define a certain kind of simplicial complex on K.

Definition 4.7. The *neighborhood complex* $\mathcal{N}(G)$ of a graph G is defined such that a set X of vertices of G is in $\mathcal{N}$ if all $x \in X$ are connected to some common vertex v.

We have that $\mathrm{conn}(\mathcal{N}(K(n,k))) = n - 2k - 1$.

Then, consider the following lemma.

Lemma 4.8. *Given a graph G,*

$$\chi(G) \geq \mathit{conn}(\mathcal{N}(G)) + 3.$$

Since the complete graph K_m is obviously m-colorable, the statement that G is m-colorable is equivalent to the statement that there is a graph homomorphism (a map $f : V(G_1) \to V(G_2)$ such that if x, y are vertices that are connected in G, $f(x)$ and $f(y)$ are connected in G_2) from G to K_m. Thus $\mathcal{N}(G)$ and $\mathcal{N}(K_m)$ are homeomorphic, so there exists a continuous antipodal map $f : S^{\mathrm{conn}(\mathcal{N}(G))} \to S^{m-2}$. By Borsuk–Ulam (vi), this means that $k+1 \leq m-2 \Longrightarrow \chi(G) \geq \mathrm{conn}(\mathcal{N}(G)) + 3$, as desired.

Finally,

$$\chi(K(n,k)) \geq \mathrm{conn}(\mathcal{N}(K(n,k))) + 3 = n - 2k + 2,$$

as desired. □

5. Fair Divisions

This section concerns problems that involve the equal division of continuous and discrete measures that are solved by Borsuk–Ulam.

First, we define some relevant language. Consider the analogous properties of length, area, and volume in one, two, and three dimensions, respectively. We can extend this idea to any n dimensions.

Definition 5.1. The *Lebesgue measure* $\lambda(S)$ where $S \subseteq \mathbb{R}^d$ is the standard assignation of size in d-dimensional space.

For the purposes of this discussion, we consider the *finite Borel measure.*

Definition 5.2. A *finite Borel measure* μ on $\mathbb{R}^d$ is the restriction of the usual Lebesgue measure to a compact subset of $\mathbb{R}^d$.

Now, the idea of antipodal points in Borsuk–Ulam lends itself neatly to the idea of dividing something fairly among two parties. Moreover, in the recurring spirit of this study, once we are able to find solutions to the continuous versions of these problems (i.e. dividing a mass), we can extend them to the combinatorially-phrased discrete analogues.

For our first problem, consider a ham sandwich consisting of the three masses of ham, cheese, and bread (disregard the anatomy of a real-life ham sandwich, which usually uses two pieces of bread, for this problem).

Two friends wish to divide this sandwich among themselves with a single planar cut such that each of them has an equal amount of each ingredient. Is this always possible?

It turns out that it is.

Theorem 5.3 (Ham sandwich). *Let $\mu_1, \mu_2, \ldots, \mu_d$ be mass distributions on $\mathbb{R}^d$. There exists a hyperplane h that divides $\mathbb{R}^d$ into two half-spaces h^+ and h^- such that*

$$\mu_i(h^+) = \mu_i(h^-)$$

for $i = 1, 2, \ldots, d$.

The example of the ham sandwich is the $d = 3$ case of the theorem. The proof is short.

Proof. For every vector $v = (v_0, v_1, \ldots, v_d) \in S^d$, we define the half-space h^+ given v such that

$$h^+(x) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : v_0 + v_1 x_1 + \cdots + v_d x_d \geq 0\}$$

Then over i we define $f_i : S^d \to \mathbb{R}^d$ such that

$$f_i(v) = \mu_i(h^+(v))$$

and $f(v) = (f_1(v), f_2(v), \ldots, f_d(v))$. By Borsuk–Ulam (i) there exists v_0 such that $f(v_0) = f(-v_0)$. Two antipodal points correspond to half-spaces, so we are done. □

Corollary 5.4 (Ham sandwich, discrete). *Let $A_1, A_2, \ldots, A_d$ be disjoint finite point subsets of $\mathbb{R}^d$ such that at most d points of $A_1 \cup A_2 \cup \cdots \cup A_d$ are contained in any hyperplane. Then there exists a hyperplane h that bisects each A_i, where being bisected is defined as*

1. *having exactly $\lfloor \frac{1}{2}|A_i| \rfloor$ points on each of h^+ and h^-, and*
2. *having at most one point of A_i on h.*

Our second problem considers the idea of splitting the multicolored beads on a necklace equally.

Definition 5.5. In $\mathbb{R}^d$, the moment curve is the locus of points with Cartesian coordinates of the form $(t, t^2, t^3, \ldots, t^d)$.

Theorem 5.6 (Necklace). *Consider an open necklace consisting of stones of d colors with an even number of stones of each color. Then we can split the necklace at d points into pieces that can be divided between two people such that each gets the same amount of stones of each color.*

Example. Consider the following necklace.

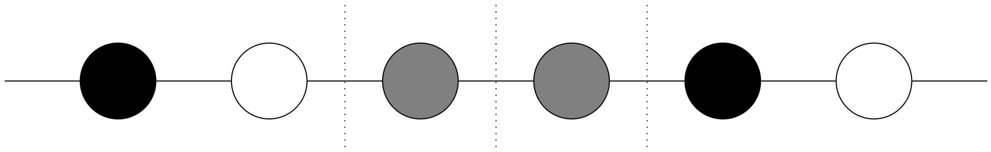


FIGURE 5. $d = 3$.

Proof. Situate the necklace in $\mathbb{R}^d$ along the moment curve $\gamma(t) = (t, t^2, t^3, \ldots, t^d)$. Define A_i to be the set of points coinciding with the locations of each bead of color i. Then by the ham sandwich theorem there exists a hyperplane h that bisects each A_i, and since $|A_i|$ is even, no stones are contained in the hyperplane. Since any h cuts the moment curve at at most d places, we are done. □

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