HOMOMESY, TOGGLES, AND ROWMOTION

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ABSTRACT. Homomesy happens in dynamical algebraic combinatorics where average value of some statistic is the same on every orbit of a combinatorial dynamical system. It was introduced and systemized by Propp and Roby, and it was seen in context of rowmotion and promotion on order ideals of posets, as can be seen in [2, 3]. This paper introduces homomesy's basic definition, gives a simple example on permutations, and then focuses deeper by looking at it through toggle-groups for order ideals. We show rowmotion on products of two chains and we demonstrate why the cardinality statistic on order ideals for these posets is homomesic. At the end, we briefly cover piecewise-linear and birational lifts of these actions.

1. INTRODUCTION

In combinatorics it's essential to understand how large families of discrete objects behave and how simple operations act on them. Often, we have an invertible map

$$\tau : S \to S$$

on a finite set S of combinatorial objects such as permutations, order ideals of a poset, Young tableaux, etc. If we iterate τ, we get orbits that may often look quite complicated.

Homomesy is the observation that, in many such systems as we described, certain statistics $f : S \to \mathbb{R}$ have the same average over every orbit of the action. At first this looks coincidental, and it doesn't make sense for a statistic to be so well behaved on each orbit when the orbit structure itself can be wild. Propp and Roby noticed that this phenomenon kept occurring, and so they named it a framework [2]. Since then, homomesy has become a key organizing idea in *dynamical algebraic combinatorics* [3, 5].

This paper will begin by giving a precise definition of homomesy and a simple example on permutations. Then we introduce toggles and the toggle group acting on order ideals of a poset. We define rowmotion and explore a fundamental homomesy result on products of two chains [2]. We also briefly describe piecewise-linear and birational analogues and mention further directions of research [1]. Throughout this paper, all sets are finite, and all maps we consider are bijections unless we explicitly state otherwise.

2. STARTING DEFINITION AND EXAMPLE

We begin with our basic definition. Let S be a finite set and $\tau : S \to S$ be a bijection. For $x \in S$, the *orbit* of x is

$$\mathcal{O}(x) := \{\tau^k(x) \mid k \in \mathbb{Z}\}$$

which is finite because S is finite. The orbits of τ partition S.

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Definition 2.1. We let S be a finite set, $\tau : S \to S$ a bijection, and $f : S \to \mathbb{R}$ a statistic. We say that the triple (S, τ, f) is **homomesic** if there exists a constant $c \in \mathbb{R}$ such that for every τ-orbit $\mathcal{O} \subseteq S$ we have

$$\frac{1}{|\mathcal{O}|} \sum_{x \in \mathcal{O}} f(x) = c.$$

In this case we say that f is **homomesic** or c-**mesic** under the action of τ.

Since S is finite and the orbits of τ partition S, we know that there is a simple global characterization of homomesy.

Proposition 2.2. *Let S, τ, and f be as we described above. Then f is homomesic under τ iff the average of f on every orbit equals the global average*

$$\frac{1}{|S|} \sum_{x \in S} f(x).$$

Proof. We write $S = \bigsqcup_i \mathcal{O}_i$ as a disjoint union of orbits. Then

$$\frac{1}{|S|} \sum_{x \in S} f(x) = \frac{1}{\sum_i |\mathcal{O}_i|} \sum_i \sum_{x \in \mathcal{O}_i} f(x).$$

If each orbit has the same average c, then $\sum_{x \in \mathcal{O}_i} f(x) = c \cdot |\mathcal{O}_i|$ for all i, so the global average is also c. On the other hand, if the average of every orbit equals the global average, then we know all orbit averages are equal to each other, so f is homomesic. ■

2.1. **A Permutation Example.** We now offer a very simple but important example. Let $S = S_n$ be the symmetric group on $\{1, \ldots, n\}$, which is viewed as permutations written in one-line notation. We define $\tau : S_n \to S_n$ by cyclic rotation

$$\tau(\sigma_1 \sigma_2 \ldots \sigma_n) = \sigma_2 \sigma_3 \ldots \sigma_n \sigma_1.$$

It is clear this is a bijection, and each orbit consists of the cyclic rotations of some word.

We then let $f : S_n \to \mathbb{R}$ be the statistic that records the position of the value n, or

$$f(\sigma) = \text{the position of } n \text{ in the one-line notation of } \sigma.$$

For example, if $n = 4$ and $\sigma = 2413$, then $f(\sigma) = 3$.

Proposition 2.3. *The statistic f is homomesic under the rotation action τ on S_n with an average value of $\frac{n+1}{2}$.*

Proof. We can fix a permutation $\sigma \in S_n$ and look at its orbit under τ. As we cyclically rotate the entries, we see that the value n visits each position $1, 2, \ldots, n$ exactly once before returning to its starting point. Thus, along the orbit $\mathcal{O}$ of σ, we have

$$\{f(x) \mid x \in \mathcal{O}\} = \{1, 2, \ldots, n\}$$

as multisets, which means that the orbit-average of f is

$$\frac{1}{n} \sum_{k=1}^{n} k = \frac{n+1}{2},$$

and is independent of the orbit.

Conversely, by symmetry, the global average of f over all of S_n is also $\frac{n+1}{2}$ since in a uniformly random permutation, each position is equally likely to contain n. Finally, by Proposition 2.2, f is homomesic with average $\frac{n+1}{2}$. ■

This example serves as a model for much more complicated situations such as a natural combinatorial action, a simple statistic, and a rigid orbit-average.

3. Toggles, order ideals, and the toggle group

Some of the best examples of homomesy arise from actions on order ideals of a poset. So, we now introduce the basic language of toggles and the toggle group, following Striker and Williams [4, 6].

Definition 3.1. Let P be a finite poset. An **order ideal** of P is a subset $I \subseteq P$ such that

$$x \in I \text{ and } y \leq x \implies y \in I.$$

We write $J(P)$ for the set of all order ideals of P.

We can think of order ideals of P as downward-closed subsets in the Hasse diagram. Whenever a point is in the ideal, everything below it must also be included in the ideal. For each element $p \in P$, we define an involution t_p on $J(P)$, called the *toggle at* p.

Definition 3.2. For $p \in P$, the **toggle** $t_p : J(P) \to J(P)$ is defined as

$$t_p(I) = \begin{cases} I \cup \{p\} & \text{if } p \notin I \text{ and } I \cup \{p\} \in J(P) \\ I \setminus \{p\} & \text{if } p \in I \text{ and } I \setminus \{p\} \in J(P) \\ I & \text{otherwise} \end{cases}$$

We are able to verify that t_p is an involution since $t_p^2 = \mathrm{id}$. The toggles generate a subgroup of the permutation group of $J(P)$.

Definition 3.3. The **toggle group** of P is the subgroup of the symmetric group on $J(P)$ generated by the toggles $\{t_p : p \in P\}$. We denote it with $\mathcal{T}(P)$.

Compositions of toggles cause many interesting dynamical systems on $J(P)$. However, one very important element of the toggle group that we will explore is *rowmotion*.

4. Rowmotion and Homomesy on Products of Two Chains

We let P be a finite poset. There are numerous equivalent ways to define rowmotion on $J(P)$. However, we use the following combinatorial description (for example see [3, 5]).

Definition 4.1. Let P be a finite poset. For an order ideal $I \in J(P)$, we define

$$\mathrm{Row}(I) := \text{the order ideal that's generated by the minimal elements of } P \setminus I$$

The map $\mathrm{Row} : J(P) \to J(P)$ is called **rowmotion**.

In other words, to get $\mathrm{Row}(I)$, we look at which elements of P can be added to I and fill in everything below them to get a new order ideal.

Cameron and Fon-Der-Flaass showed that we can also express rowmotion as a product of toggles taken in a certain order, or as a distinguished element of the toggle group $\mathcal{T}(P)$. The later work of Striker and Williams places this in a broader framework and relates rowmotion to another toggle-group element called promotion [4, 6]. This toggle-group perspective is essential in understanding and many homomesy proofs.

4.1. **Products of Two Chains.** The posets where homomesy for rowmotion was first properly studied are products of two chains. We fix positive integers a, b and let

$$P = [a] \times [b]$$

with partial order

$$(i, j) \leq (i', j') \iff i \leq i' \text{ and } j \leq j'.$$

We can picture P as an $a \times b$ grid of lattice points ordered southwest to northeast.

The order ideals of $[a] \times [b]$ can be visualized as Young diagrams or lattice points that fit inside an $a \times b$ rectangle. Each order ideal translates to a monotone path from the southwest corner to the northeast corner, and adding or removing a box means toggling an element of the poset. A fundamental homomesy result of Propp and Roby explains that the **cardinality** statistic on $J([a] \times [b])$ is homomesic under rowmotion.

Theorem 4.2 (Propp–Roby)**.** *Let* $P = [a] \times [b]$ *and let* $\mathrm{Row} : J(P) \to J(P)$ *be rowmotion. Consider the statistic*

$$f(I) = |I|$$

with $I \in J(P)$*. Then* $(J(P), \mathrm{Row}, f)$ *is homomesic, meaning the average size of an order ideal along any rowmotion orbit equals to*

$$\frac{1}{2} \cdot |P| = \frac{ab}{2}.$$

In other words, if we follow an orbit of rowmotion and take note of the sizes $|I|$ of the order ideals we see, the average will always be $\frac{ab}{2}$, regardless of whichever orbit we started with. We only sketch the proof here. For the full proof and details see [2, 3]. The main idea is to track how often each element $p \in P$ occurs in an order ideal as we move along a rowmotion orbit.

We first fix an element $p \in P$ and consider the indicator statistic

$$f_p(I) = \begin{cases} 1 & \text{if } p \in I \\ 0 & \text{otherwise} \end{cases}$$

We can write the size statistic as

$$f(I) = |I| = \sum_{p \in P} f_p(I).$$

Thus, to prove that f is homomesic, it is enough to show that each f_p has average $\frac{1}{2}$ on every orbit. Summing over p will then give us an average of $\frac{|P|}{2}$.

Propp and Roby analyze how the set p belongs in the current ideal changes when we apply rowmotion. In the toggle-group description, rowmotion can be written as a product of toggles that is taken in a linear extension of the poset. For $P = [a] \times [b]$, there is a high degree of symmetry since for each element p, there is also a complementary element p' such that along an orbit, p and p' appear in a balanced type of way. This then leads us to the fact that over the orbit, p is present exactly half of the time.

We can formalize this argument in several ways. It can be done through explicit combinatorial pairing along orbits, through piecewise-linear or birational lifts [1], or through representation-theoretic interpretations. Of course, the outcome is the same in that the average of f_p on every orbit is $\frac{1}{2}$, so the average of $|I|$ is $\frac{|P|}{2}$ on every orbit, proving Theorem 4.2.

5. Piecewise-linear, Birational Rowmotion and Future Directions

A very shocking aspect of rowmotion and homomesy is that they extend beyond just the combinatorial setting. Einstein and Propp [1] constructed piecewise-linear and birational analogues of rowmotion that act on real-valued labelings of a poset. In these settings, we see how combinatorial statistics are replaced by rational functions, and the homomesy phenomenon continues.

Roughly speaking, we can look at the combinatorial rowmotion on $J(P)$ as a tropical shadow of more general dynamical systems on $\mathbb{R}^P$ or on fields of rational functions. The birational version of rowmotion satisfies identities that, when specialized or tropicalized, recover the combinatorial homomesies on order ideals. This brings dynamical algebraic combinatorics to cluster algebras, total positivity, and other parts of algebraic geometry, allowing for some of the most elegant connections.

There are many more examples and generalizations of homomesy. For example, there is homomesy of statistics on standard Young tableaux under promotion and evacuation. A more practical one is Homomesy for actions on parking functions, plane partitions, and alternating sign matrices. One of the most surprising is homomesy's connections to representation theory and to cyclic sieving phenomena. Surveys such as [3, 5] give a more general look onto the field and lists many more open problems.

Homomesy shows us how even when the orbit structure of a combinatorial dynamical system is complicated, certain statistics are able to behave rigidly, their average remaining constant across all orbits. Look at this using the toggle-group and the study of rowmotion on products of chains gives us a clean and accessible entry point into this phenomenon. Also, piecewise-linear and birational lifts show the deeper algebraic structure at its work. More broadly, homomesy serves to show how ideas from dynamics, algebra, and combinatorics intersect. A simple definition leads us to many rich examples, unexpected symmetry, and ongoing research directions.

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