# KEPLER TOWERS (CATALAN OBJECTS)

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Abstract

In this paper, I will dive into the deep world of Catalan numbers, exploring examples I find interesting and why those are Catalan objects. I will also have a special focus on Kepler Towers, one specific type of Catalan object that seems to be relatively unstudied and has a nonobvious bijection with Dyck words, another Catalan object.

## 1. Introduction

There are hundreds of objects that can be counted by the Catalan numbers. One can show they are Catalan objects by either relating it back to the Catalan recurrence, or showing that their fundamental structure ends up being a bijection to a Catalan object we already know of.

## 2. Background

**Definition 2.1.** Catalan Numbers
Define the $n$th Catalan number to be $C_n = \frac{1}{2n+1}\binom{2n+1}{n}$.

**Definition 2.2.** Dyck Paths
Recall that a Dyck Path is a path of right and up steps from $(0, 0)$ to $(n, n)$ without going above the line $y = x$ (but able to touch it). We can count these by counting all paths of right and up steps that go to $(n, n)$, then subtracting by the paths to $(n - 1, n + 1)$, since those are bijections to non-Dyck paths by finding the first time that path goes above (not touches) $y = x$, then swapping all up steps to right steps and right steps to up steps from that point on. This yields the result of $\frac{1}{2n+1}\binom{2n+1}{n}$, equal to $C_n$.

Occasionally, these will also be represented as counting the paths using $(+1, +1)$ steps and $(+1, -1)$ steps to get to $(2n, 0)$ without going under $y = 0$, but these are essentially the same. These can also be represented as Dyck words, a sequence $d_0, d_1, \ldots d_{2n-1}$ of $\pm 1$s that sum to 0 and have partial sums $d_0 + d_1 + \ldots + d_k$ that are never under 0. The height of these Dyck words or modified Dyck paths is the maximum partial sum, or the height of the highest point in the path. We will use this definition when we cover Kepler towers later in the paper.

Now that we know Dyck paths count Catalan numbers, we can create a recursive relationship for Dyck paths that will hold for Catalan numbers as well.

**Definition 2.3.** Catalan Recurrence

$$C_{n+1} = C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \cdots + C_{n-1}C_1 + C_nC_0 = \sum_{k=0}^{n} C_kC_{n-k}.$$

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We can count the Dyck paths to $(n+1, n+1)$ by combining earlier Dyck paths in order to prove this relationship. These paths must hit the line $y = x$ after $(0,0)$ at least once, whether it be at $(n+1, n+1)$ or earlier. When it first does so at some $(k+1, k+1)$, where $0 \leq k \leq n$, the path it took going from $(0,0)$ to $(k+1, k+1)$ must have started with a right step and ended with an up step, and the rest of the path couldn't have touched $y = x$ at all, so it is a like a Dyck path of $k$ but shifted down 1 unit. The rest of the path continues to be a Dyck path, so it is the Dyck path of $n-k$. Since we can have any $k$ where $0 \leq k \leq n$, we add the number of paths for each case of $k$ to get the total of

$$C_{n+1} = C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \cdots + C_{n-1}C_1 + C_nC_0 = \sum_{k=0}^{n} C_kC_{n-k}.$$

## 3. Examples

Let's briefly go over some more common Catalan objects.

**Definition 3.1.** Parenthesizations
Define a parenthesization as an expression with $n+1$ symbols, such as letters, where pairs of parentheses following normal parentheses rules contain exactly two symbols or symbol groups (another parenthesization), and where the expression has two outer parentheses. For example, $(a(bc))$ is a valid parenthesization, while $a(bc), (abc), (ab)(c$ are not.

It can be shown these are bijections to Dyck paths, by deleting the last symbol and all right parentheticals, then converting left parentheticals to right steps and symbols to up steps.

**Definition 3.2.** Binary Trees
Define a binary tree to be a graph with no cycles where every vertex has at most 2 children, which are vertices farther from an assigned root than its parent it is connected to."

This can be quickly related to the Catalan sequence in two ways. First, it can be shown its recurrence relation is the same, as well as its initial term. To count the binary trees of $n+1$ vertices, $B_{n+1}$, split the tree through its root, making two binary trees (that could potentially be of 0 vertices). The left side will have some $k = 0, 1, 2, \ldots n$ vertices, and the right would then have $n + 1 - k - 1 = n - k$ vertices. Summing over all $k$,

$$B_{n+1} = B_0B_n + B_1B_{n-1} \ldots = \sum_{k=0}^{n} B_kB_{n-k}.$$

Finally, since $B_0 = 1$ (1 way to have no vertices), we know that $B_n$ is the same sequence as $C_n$.

Similar to binary trees, triangulations of an $(n+2)$-gon into $n$ triangles, 231-avoiding permutations of $[n]$, and 123-avoiding permutations of $[n]$ can be shown to be Catalan objects by dividing the object somewhere in the middle into two objects with a smaller $n$, then summing those up for all possible splits (0 to $n-1$) to get the number of objects for $n$ (As well as showing there is only 1 object for $n = 0$).

## 4. Kepler Towers

One interesting object that Richard Stanley mentions in his famous list of Catalan Objects [Sta13] is the Kepler Tower. In this structure there are:

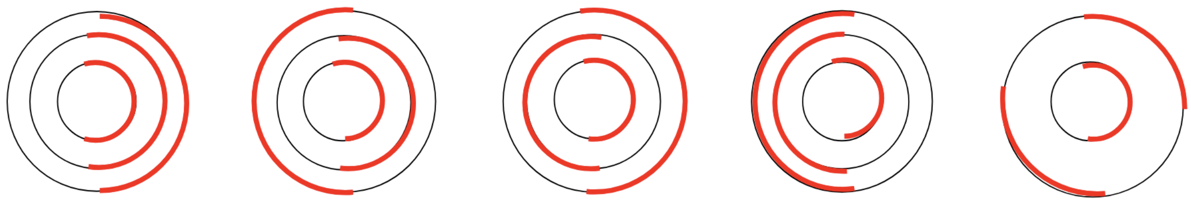

**Figure 1.** 3-brick Kepler Towers

- $n$ bricks
- $k$ walls; each is a set of concentric circles, known as rings. Let ring $n$ of a wall be the ring that is $n$th closest to the center compared to the other rings in the wall.
- Concentric circles in the $i$th wall divided into $2^i$ arcs; arc 1 will start from the top of the circle and goes clockwise from there, arc 2 will start from the right endpoint of arc 1 and goes clockwise, etc., just like a clock. (Exact index assignment for arcs varies between sources, but it doesn't really matter; some people also use $2^i$-gons instead of concentric circles)
- Bricks placed onto these arcs, that extend an arbitrarily small distance past the arc it occupies (useful later), can't be adjacent to one another
- Innermost circle in each wall automatically has bricks on arcs $1, 3, 5, 7 \dots 2^i - 1$
- If there is an brick $B$ on ring $j$ that isn't on innermost circle of a wall, ring $j-1$ has a brick $B'$ such that there is a ray from the center that intersects both. Essentially, if there is a brick on arc $m$ on ring $j-1$, there must be a brick at either arc $m-1$, $m$, or $m+1$, wrapping around if needed because we are using circles.
- No empty walls, nor walls with no rings

**Theorem 4.1.** *The number of Kepler towers that use $n$ bricks is equal to $C_n$.*

Don Knuth, a computer science professor, describes a non-obvious approach to creating bijections between Kepler towers and Dyck words, thus proving this theorem. Although this bijection doesn't intuitively make too much sense, it can be proved that it does actually make a bijection. [Knu05]

Recall that Dyck words are a sequence of $d$s, that are $\pm 1$s, starting at $d_0$ as this is a computer science paper. Append $d_{2n} = -1$ to the sequence; now the total sum is $-1$. Then, read the Dyck word into chunks that will represent walls, beginning the split of the $k$th wall after the moment the partial sum first hits $2^k - 1$. For example, if the partial sum hits $2^3 - 1 = 7$ at $d_0 + \ldots + d_p = 7$, then the 3rd wall begins at $d_{p+1}$. Notice how since $d_0$ is always 1, we always start building the first wall with $d_1$. This makes sense, since $d_0$ conveys no information as it is always 1, so it shouldn't have any impact on the Kepler tower construction unless it conveys information we would already know.

Then, we must read the sequence of each wall to generate the bricks it will contain. Let the outermost wall, the $j$th wall, which has some $r = 2^j$ segments be known as an $r$-path. This must have a sequence of $\pm 1$s whose partial sums remain strictly less than $r$ and strictly greater than $-r$ but whose total sum is $-r$. Since the wall starts after the sum hits $2^j - 1$, and the total sum up until $d_{2n}$ is -1, the partial sum from the wall start to its end (recall that this end will also be the end of the Dyck word since this wall is outermost), is $-2^j = -r$. If the partial sum hits $-r$ before the total does, then the path had to have ended at that point, so there is a contradiction. If the partial sum hits $r$ at some point, then the total sum at that point will be $2^{k+1}-1$, so there is another wall, and there is once again a contradiction.

Define a dual r-path to be an r-path whose total sum is $r$ instead of $-r$. These appear for the sequence for some $k$th wall that is not outermost, and it is also simple to see that the partial sums remain strictly less than $r$ and strictly greater than $-r$. From now on, the terms wall and path are essentially interchangeable.

Let's actually begin building the walls now. First, the bricks in the innermost ring, ring 1, are predetermined by definition, so those are automatically generated instead of by Dyck word; if our algorithm makes it so the Kepler tower includes some wall, its assumed that the innermost already has bricks on all odd segments.

Then, we work on adding the bricks that are not in the innermost ring of each wall, reading each wall from left to right and looking at the partial sum from the start of the wall to the Dyck letter we are on currently. When this partial sum changes from 0 to 1, change into "downward mode," in which a brick drops into segment $s$ when the partial sum decreases from $s$ to $s-1$. When the partial sum changes from 0 to $-1$, change into "upward mode," where a brick drops into segment $s$ when the partial sum increases from $s-r-1$ to $s-r$. Essentially, when positive, we are in downward mode, and when negative, we are in upward mode. In both cases, a brick dropping into a segment means that it gets placed into the uppermost ring for which it has support from below.

This is a lot to process, so let's go through an example before we show why the algorithm is a bijection. Consider the following Dyck word of length $2n = 14$:

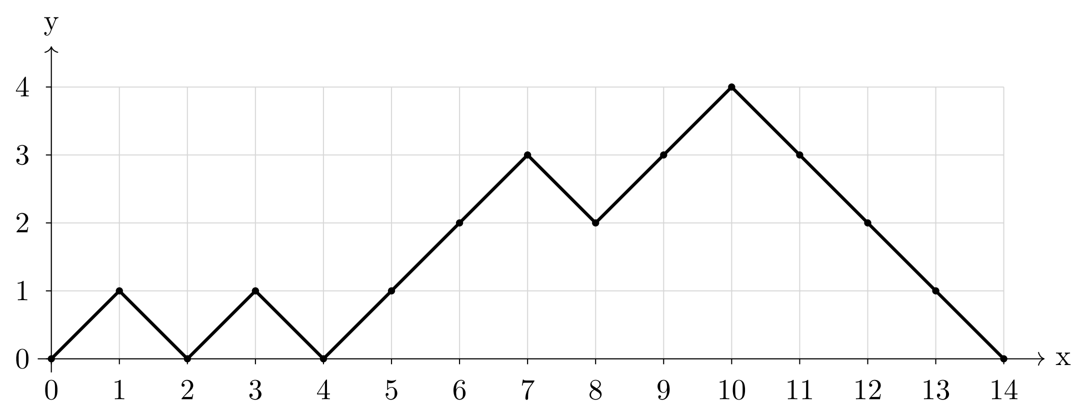


**Figure 2.** Dyck word for sequence + − + − + + + − + + − − − −

First, we need to append $d_{2n} = -1$, and then split this path into our walls. Let's get the partial sums:

| $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ | $d_7$ | $d_8$ | $d_9$ | $d_{10}$ | $d_{11}$ | $d_{12}$ | $d_{13}$ | $d_{14}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 2 | 3 | 2 | 3 | 4 | 3 | 2 | 1 | 0 | $-1$ |

This hits $2^1 - 1 = 1$ at $d_0$ and $2^2 - 1 = 3$ at $d_6$, so wall 1 consists of $d_1$ to $d_6$, and wall 2 consists of $d_7$ to $d_{14}$.

Next, we look at the partial sums concerning each individual wall:

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ | $d_7$ | $d_8$ | $d_9$ | $d_{10}$ | $d_{11}$ | $d_{12}$ | $d_{13}$ | $d_{14}$ |
| 1 | $-1$ | 0 | $-1$ | 0 | 1 | 2 | $-1$ | 0 | 1 | 0 | $-1$ | $-2$ | $-3$ | $-4$ |

Notice how these follow our results concerning r-paths (wall 2) and dual r-paths (wall 1).

Finally, let's find out which segments we are to place bricks in, then we can add in the given innermost rings and we are done. We "translate" the partial sums of each wall using our upward/downward mode method:

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ | $d_7$ | $d_8$ | $d_9$ | $d_{10}$ | $d_{11}$ | $d_{12}$ | $d_{13}$ | $d_{14}$ |
| 1 | $-1$ | 0 | $-1$ | 0 | 1 | 2 | $-1$ | 0 | 1 | 0 | $-1$ | $-2$ | $-3$ | $-4$ |
| down | up | 2 | up | 2 | down | | up | 4 | down | 1 | up | | | |

We have to place bricks in arcs 2 and 2 in wall 1, and in arcs 4 and 1 in wall 2, in those orders. Recall that a brick dropping into a segment means that it gets placed into the uppermost ring for which it has support from below. Ring 1 of wall 1 has a brick in arc 1, and ring 1 of wall 2 has bricks in arcs 1 and 3. Bricks cannot be adjacent or occupy the same spot, so for wall 1, we must place bricks into arc 2 in ring 2 and arc 2 of ring 3. Similarly, for wall 2, we must place bricks into arc 4 in ring 2 and arc 1 of ring 3. Now, we have finished our Kepler tower!

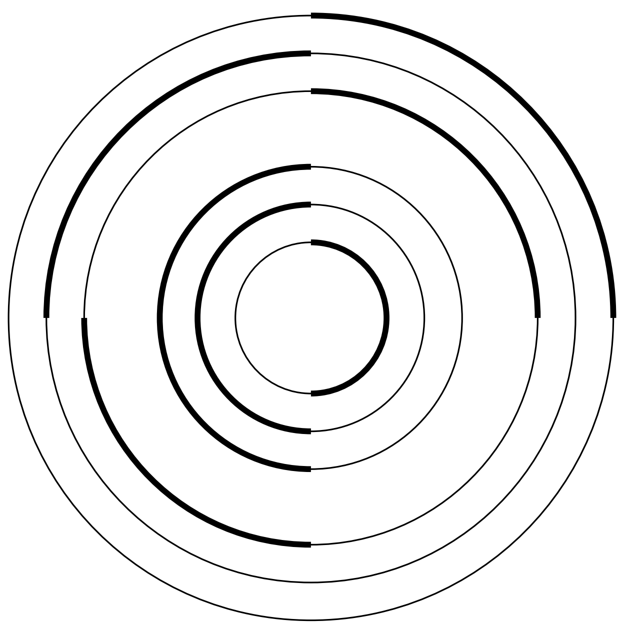

**Figure 3.** Our Final Kepler Tower

## 5. Studying the Bijection

Now, we can more easily delve into why the bijection works. First, we'll show that a Dyck word of $2n$ length always maps to a Kepler tower of $n$ bricks, then show that our algorithm creates a bijection because each Dyck word of length $2n$ maps to exactly one Kepler tower with $n$ bricks and each Kepler tower with $n$ bricks maps to exactly one Dyck word of length $2n$.

**Theorem 5.1.** *When creating a Kepler tower from a Dyck word, the outermost wall, or the non-dual path, places a brick for each* $+1$*, and the other walls, or the dual paths, place a brick for each* $-1$*, albeit their positions compared to the path don't exactly align.*

*Proof.* Although Knuth doesn't prove this, it is not too difficult to do ourselves. Let's start by looking at the case where there is a $+$ in the non-dual path. If the partial sum is negative at that point, great! We have added a brick, since we are in upward mode. However, when we are in downward mode, meaning the partial sum is nonnegative, we don't necessarily add a brick at the moment. But, since our goal total sum is negative, as this is a non-dual r-path, a $+$ means we need to have a corresponding $-$ at some point when the partial is positive in order to make the partial negative and reach the goal, thus also adding a brick. Similar logic applies for the dual r-paths, but since the goal total sum is positive, a $-$ while the partial is positive adds a brick, while a $-$ while the partial is nonpositive means we need a corresponding $+$ while it is still nonpositive, adding a brick as well. ∎

Now, we prove the first part of a proof that would show a bijection:

**Lemma 5.2.** *Each Dyck word of length* $2n$ *maps to exactly one Kepler tower with* $n$ *bricks.*

*Proof.* Let wall/path $w$ have $n_w \pm 1$'s inside. When this is a dual path, its total sum is $2^w$, so there are $2^w$ more $+1$s than $-1$s. This means that the number of $-1$s in this path is $\frac{n_w-2^w}{2}$, so there are $\frac{n_w-2^w}{2}$ bricks in this specific wall, not including the automatically generated ones. When we do so, since there are $2^{w-1}$ automatically generated bricks in ring 1, the total brick count of this wall becomes $\frac{n_w}{2}$. In the case of $w = 0$, notice that there are no automatically generated bricks, so there will be $\frac{1-2^0}{2} = 0$ bricks, as expected. Similarly, it can also be shown that when $w$ is a non-dual wall, there are $\frac{n_w}{2}$ bricks.

Now, we know that we place a total of $\frac{\sum_{w=1} n_w}{2}$ bricks. We know $\sum_{w=0} n_w = 2n+1$, since this represents the length of the entire sequence and we go from $d_0$ to the $d_{2n}$ we appended. Since $n_0 = 1$, $\frac{\sum_{w=1} n_w}{2} = n$, so our Dyck word of length 2n will map to a Kepler tower of $n$ bricks.

We still should show that the Dyck word doesn't map to two or more Kepler towers, but our algorithm does not incorporate any sort of choosing, so we know this is true. ∎

We also need the second part:

**Lemma 5.3.** *Each Kepler tower with* $n$ *bricks maps to exactly one Dyck word of length* $2n$*.*

There will be no formal proof for this, but the algorithm we used can be reversed, going from outer ring to inner ring for each wall; this will also have no forks or choices, so we know all Kepler towers map to exactly one Dyck word. Finally, we can reverse our logic in Theorem 5.1 to show that the mapping for a Kepler tower with $n$ bricks maps to a Dyck word of length $2n$. Thus, we have proved Theorem 4.1: The number of Kepler towers that use $n$ bricks is equal to $C_n$.

## 6. Closing Thoughts

Kepler Towers were quite interesting to study, because it was completely nonobvious that they were counted by Catalan numbers, even once you figure out the bijection algorithm.

Trying to count them by splitting them into two smaller Kepler Towers also gets confusing because the automatically generated ring 1 adds in a varying amount of extra bricks. Perhaps a different bijection or intuitive interpretation may come for these one day, but for now they remain confusing yet interesting.

## References

[Knu05] Donald E. Knuth. Three catalan bijections. Technical report, Institut Mittag-Leffler, Stanford University, 2005. Available at `https://cs.stanford.edu/~knuth/papers/tcb+.pdf`.

[Sta13] Richard P. Stanley. Catalan addendum. Technical report, Massachusetts Institute of Technology, 2013. Available at `https://math.mit.edu/~rstan/ec/catadd.pdf`.