Differential Posets

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Abstract

After a quick review of terminology, we define an r-differential poset. An r-differential poset is a poset P that satisfies three properties: P is locally finite, graded, and has a $\hat{0}$ element, whenever $x, y \in P$ for $x \neq y$ and there are k elements covered by both x and x, there are also k elements covering both x and y, and if $x \in P$ covers k elements of P, then x is covered by $x + r$ elements of P. If P is r-differential for some r, we say that P is a differential poset. We then familiarize ourselves with the connection to calculus by giving a different, significantly longer definition. This new definition pairs each differential poset with two operators. After proving the two definitions are equivalent, we look at multiple examples of differential posets: the Young poset and the r-Fibbonacci poset. Next, we will see some results regarding differential posets, including 1.2 - 1.4 from R.P Stanley's first paper on the subject. Finally, we look at a few open problems regarding differential posets.

1 Introduction

Definition 1. *A poset P is some set paired with a binary relation $\leq$ such that $x \leq x$ for all $x \in P$, $x \leq y$ and $y \leq x$ together imply $x = y$, and $x \leq y$ and $y \leq z$ together imply $x \leq z$.*

Definition 2. *Let P be a poset. We say that $x \in P$ covers $y \in P$ if $y \leq x$ and there is no $z \in P$ such that $y \leq z \leq x$.*

Definition 3. *Let P be a poset. We say that P is locally finite if every interval $\{z | x \leq z \leq y\}$ for $x, y \in P$ is finite.*

Definition 4. *Let P be a poset. We say that P is graded if there is an order-preserving injective function from P to $\mathbb{Z}$.*

Definition 5. *Let r be some positive integer. The poset P is r-differential if P is locally finite, is graded, and has a minimal element $\hat{0}$.*

1. *If $x, y \in P$, $x \neq y$, and there are k elements of P covered by both x and y, then there are also k elements covering both x and y.*

2. *If $x \in P$ covers exactly k elements of P, then x is covered by exactly $k + r$ elements of P.*

3. *If P is r-differential for some r, we say P is a differential poset.*

Suppose for the sake of contradiction that $k \geq 2$. Then there are two elements of P that have at least two elements covering both of them, hence these two new elements have $k \geq 2$. We continue to repeat this process forever, contradicting the fact that there is a $\hat{0}$ element.

Definition 6. *Let P be a locally finite poset such that $C^{-}(x)$ (the set of elements of P covered by x) and $C^{+}(x)$ (the set of elements of P covering x) are finite for all $x \in P$. Let $V \supseteq P$ be a vector field with P as a basis. We define functions U and D as:*

$$Ux = \sum_{y \in C^{-}(x)} y$$

$$Dx = \sum_{y \in C^{+}(x)} y$$

We extend U and D to all of K to get linear transformations $U : V \to V$ and $D : V \to V$.

Definition 7. *Let P be a locally finite graded poset with a $\hat{0}$ element, such that every rank has finitely many elements. We say the P is r-differential if $DU - UD = rI$ for some K.*

Let $\mathbb{R}$ be the typical poset on the real numbers. In fact, this is a toset, or totally ordered set. These are posets in which, for every x, y, either $x \leq y$ or $y \leq x$. Let $\mathbb{R}^{\mathbb{R}}$ be the poset of functions on the real numbers, in which $f \leq g$ if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Now, let the operator U multiply by x^r and let the operator D give the nth derivative with respect to x. Now we have $DUf = \frac{d}{dx} x f(x) = f(x) + x f'(x) = If + UDf$ for $r = 1$. This is equivalent to $DU - UD = rI$ for $r = 1$, implying $\mathbb{R}^{\mathbb{R}}$ is 1-differential. We now see that 1-differential posets are a generalization of calculus, and r-differential posets are a further generalized form.

Theorem 1. *Definition 1 and Definition 3 are equivalent.*

Proof. Let $x \in P$. Now $DUx = \sum_{y} c_y^{+} y$ where $c_y^{+} = \#(C^{+}(x) \cap C^{+}(y))$, and $UDx = \sum_{y} c_y^{-} y$ where $c_y^{-} = \#(C^{-}(x) \cap C^{-}(y))$. So $DU - UD = rI$ if and only if $\#(C^{-}(x) \cap C^{-}(y)) = \#(C^{+}(x) \cap C^{+}(y))$ and $C^{-}(x) + C^{-}(x)$ for all $x \neq y \in P$. But these are the exact conditions for P to be r-differential. □

2 Examples

Some familiar posets turn out to be differential. One easy example, originally formulated by Alfred Young, is the Young lattice, consisting of all integer partitions ordered by inclusion of Ferrers diagrams or Young diagrams.

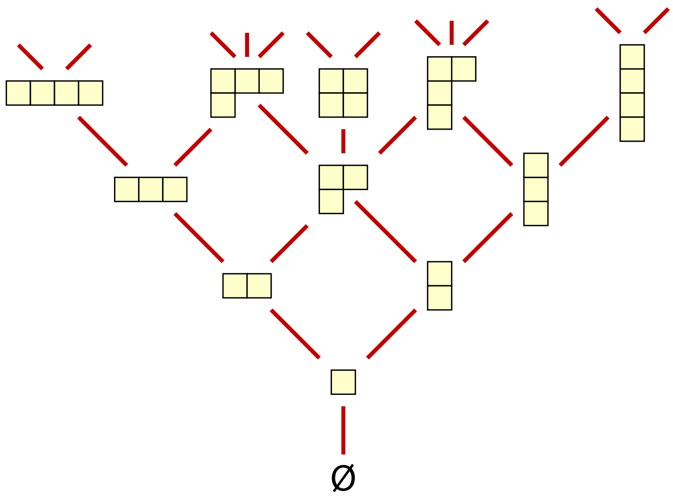

Figure 1: Young's Lattice

Proposition 1. *Let L be a lattice satisfying (1) and (3) from Definition 1. Now, L is r-differential if and only if L is modular.*

Proof. G. Birkhoff observed in *Lattice theory* that a locally finite lattice is modular if and only if the following condition is satisfied: If $x \in P$ covers exactly k elements of P, then x is covered by exactly $k + r$ elements of P. But this is (2) from Definition 1. □

The Young lattice, which satisfies (1) and has been observed by Richard P. Stanley to satisfy (3), is distributive, and thus modular. This means it is r-differential. Another example, the $Z(r)$ poset, is defined as the set of (finite) words in some alphabet $A_r = \{1_1 \ldots 1_r, 2\}$. w_1 covers w_2 if we obtain w_2 either by changing the last 2 in w_1's initial string of 2s to a 1_i or by deleting the first 1_i from w_1.

Theorem 2. *$Z(r)$ is r-differential.*

Proof. The $\hat{0}$ element is the empty word, and $Z(r)$ is graded by the sum of the "types" of the letters, where the type of 1_i is 1 and the type of 2 is 2. There are finitely many strings of each grading, so $Z(r)$ is locally finite.
We will now check condition 2. Take $x, y \in Z(r)$ so that x and y cover a unique element z. We let z consist of $k \geq 0$ consecutive 2s followed by a 1_i and then the string $s \in Z(r)$. It's possible that z does not contain any 1_i, but this changes very little.
We have two cases now:

1. $x = 2^{k+1}s$ and $y = 2^{\ell}1_j2^{k-\ell}1_is$
2. $x = 2^m1_p2^{k-m}1_is$ and $y = 2^{\ell}1_j2^{k-\ell}1_is$

In either case, we take $w = 2^{k+1}1_i s$ so that w is the unique element covering both x and y. Similarly, if there is an element covering x and y, we can construct an element covered by both. Now we check condition 3. If the initial string of 2s of x has length k, we could have obtained this initial string by replacing any 1_i in any of these positions with a 2, in kr ways; if the initial string is followed by some 1_i, then there's one more way we could have obtained x, namely from inserting this 1_i. Hence there are either kr or $kr+1$ elements covered by x, depending on whether the string has any 1s or not. To obtain strings covering x, we can place a 1_i between any two 2s or at the beginning or end of the string of 2s, in $(k+1)r$ ways, and if x has any 1s, we can also replace the initial 1 with a 2. Hence there are either $(k+1)r$ or $(k+1)r+1$ words covering x. Thus $Z(r)$ is r-differential. □

3 Concepts

There are many useful terms regarding differential posets. Among these are the r-differential poset up to rank n and the Hasse walk.

Definition 8. *Let P be a finite graded poset of rank n that satisfies the first two conditions for being an r-differential poset, and also satisfies the third for any element x of rank less than n. We call such a poset an r-differential poset up to rank n.*

Definition 9. *A Hasse walk of length n on a poset P is a sequence $x_0 \dots x_n$ of elements of P so that for each i with $0 \le i \le n-1$, $x_i + 1 \in C^+(xi) \cup C^-(xi)$.*

Theorem 3. *If P is an r-differential poset, then $DP = (U+r)P$.*

Proof. If $DP = \sum a_x x$, then $a_x = \#C^+(x)$. If $(U+r)P = \sum b_x x$, then $a_x = r + \#C^-(x)$. The result follows from the third property in the definition of r-differential. □

Corollary 1. *If P is an r-differential poset, then for any $f(U) \in K[[U]]$ we have $Df(U) = rf'(U) + f(U)D$. Moreover, if $f(U)$ defines an element of $\mathrm{End}(\hat{K}P)$, we have $Df(U)P = (rf'(U) + (U+r)f(U))P$.*

Proof. By linearity and continuity, it suffices to assume $f(U) = U^n$, $n > 0$. The proof is then straightforward. □

4 Open Problems

Problem 1. *Characterize all r-differential posets.*

D. Wagner has described a very general method for constructing differential posets that make it unlikely that Problem 1 has a reasonable answer. The following special case of Wagner's construction suffices to show that, for each r, there are infinitely many irreducible r-differential posets: Let P be a graded poset of rank n. Define the reflection extension P^+ of P to be the poset of rank $n+1$ which coincides with p for ranks $\leq n$ and has an element $x^* \in P^+_{n+1}$ for each $x \in P_{n-1}$, with the cover relations x^* covers $y \in P_n$ if y covers $x \in P_{n-1}$. Define $E_r(P)$ be the poset obtained from P^+ by adjoining r additional elements above each element $x \in P_n$.

If P satisfies properties (1) and (2) from the definition of r-differential for x and y with rank $< n$, and property (3) for x of rank $< n$, then we call P a partial r-differential poset of rank n. Note that all partial r-differential posets of rank n are r-differential posets up to rank $n+1$, but the converse is not true.

References

[1] Simon Rubinstein-Salzedo *Differential Posets*.

[2] Richard P. Stanley (1988) *Differential Posets*, Journal of the American Mathematical Society, Vol. 1, No. 4.