An Brief Summary of Topological Combinatorics

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December 4, 2025

1 Overview

The aim of this paper is to demonstrate the use of topological methods as a viable way of problem-solving in combinatorics. The most impressive result of this is a proof of the Kneser Conjecture that involves the Borsuk–Ulam Theorem. This paper will present a simplified version of the proof involving the distrbution of points along an m-dimensional sphere.

First, the idea is to motivate a connection between combinatorics and topology by presenting simplical complexes as both a graph-theoretical and a topological object.

2 Simplical Complexes

Intuitively, a simplical complex is a topological object formed by gluing together building blocks, which may be lines, points, faces, or polyhedra, in a way that follows certain intuitive rules. These building blocks are called "simplices." The idea is to topologically represent shapes and objects for the purposes of geometric modeling and interpretations, but from a purely combinatorial perspective, their aim is to emphase such ideas as connectivity and invariance.

Definition 2.1. A *simplex* is an object designed to generalize the notion of a triangle or tetrahedron to arbitrary dimensions. It represents the "simplest" possible polytope in each dimension.

So, a 0-dimensional simplex is a point, a 1-dimensional simplex is a line segment, a 2-dimensional simplex is a triangle, a 3-dimensional simplex is a tetrahedron, and so on.

More specifically, a k-simplex is a k-dimensional polytope that is the convex hull of its vertices, and are equivalent up to their dimension. So, simplices can be the basic building blocks of geometric constructions, functioning as lines, faces, and polyhedra. However, we want to be able to put simplexes together into a coherent object, to use them as building blocks. For this, we must present another definition.

Definition 2.2. A *simplical complex* is an object consisting of a set of simplices and a set of "faces," or convex hulls that are, themselves, simplices, formed by these simplices, such that the nonempty intersection of two simplices is a face of both simplices.

More general spaces like CW complexes tend to supercede these, but the connection between simplical complexes and graphs is clear. In fact, we can redefine simplical complexes to give a more direct connection to graphs. The original definition implies heavily some geometric significance, but we can define the *abstract simplical complex*, which removes this significance.

Definition 2.3. An *abstract simplical complex* is a family of sets that is closed under taking subsets; i.e. every subset of a set in the family is in the family.

This aligns with the definition for a regular simplical complex, as every subset of objects in a simplical complex is either a simplex or a face.

Now that we have an abstract definition for a simplical complex, it becomes clear how we can associate a simplical complex to a graph.

Definition 2.4. The *neighborhood complex* of a graph G, denoted $N(G)$, is the simplical complex formed by the subsets of the neighborhoods of all vertices in G. (The neighborhood of a node in a graph is the set of all points adjacent to the node, including the node itself.) Each point in the simplical complex is a subset of vertices that are neighbors in the original graph.

The primary use of neighborhood complexes in combinatorics is in graph colorings, as the connectivity of a neighborhood complex can provide a lower bound on the chromatic number on a graph. We will see an example of this in section 4. For now, we will view an example of topological methods being used to solve a coloring problem of a different nature.

3 Hex

In the game Hex, two players play on a finite grid of hexagons. One colors a hexagon red every turn, and the other blue. Once every tile is colored, the red player wins if there is a path of red hexagons connecting the top and bottom of the grid, and the blue player wins if there is a path of blue hexagons connecting the left and right of the grid.

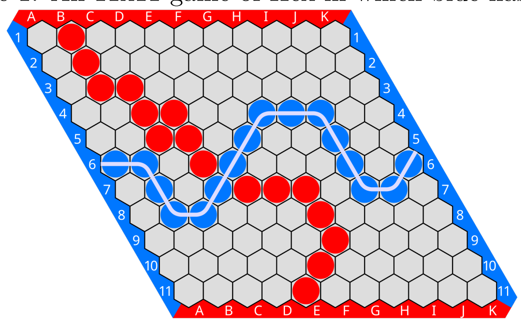
To demonstrate an example of topological methods being pertinent to combinatorial problem solving, we will demonstrate how Hex can be proven to never have a draw. Clearly, the game can have at most one winner, so it suffices to prove that it must always have a winner. To prove this, we will first invoke a theorem in algebraic topology that will go without proof.

Theorem 3.1 (Brouwer's Fixed Point Theorem). *Let B^2 be a unit disk in $\mathbb{R}^2$, and $f : B^2 \to B^2$ a continuous map. Then, there exists $x \in B^2$ such that $f(x) = x$.*

Essentially, such a mapping on the unit disk must have at least one fixed point.

Theorem 3.2. *The game Hex always has a winner.*

Figure 1: An 11x11 game of Hex in which blue has won.



Proof. We can consider a Hex board as a graph with blue of red hexagons as vertices and common sides as edges, giving us a graph in which a path from left to right of blue vertices constitutes as a win for blue, and a path from top to bottom of red verices constitutes as a win for red. Now, for the sake of contradiction, assume there are no winners in a completed game of Hex. Define the following:

- R_0: The set of red vertices which can be reached from a path connected to the bottom
- R_1: Red vertices not in R_0
- B_0: Blue vertices which can be reached from the left side by a green path
- B_1: Blue vertices not in B_0.

Additionally, define $e_1(v)$ and $e_2(v)$ as a rightward shift one vertex (parallel to the top and bottom) and an upward shift one vertex (parallel to the left and right) respectively. Now, we can define the following function:

$$f(x) := \begin{cases} e_2(v) & v \in R_0 \\ e_2^{-1}(v) & v \in R_1 \\ e_1(v) & v \in B_0 \\ e_1^{-1}(v) & v \in B_1 \end{cases}$$

This function should be well-defined by our assumption; it should not lead to any vertex shifts that go off the board. Now, consider any triangle T on the board G with vertices v_1, v_2, and v_3 serving as the convex hull of the triangle. Every point x in the triangle can be written uniquely as $x = \sum x_i v_i$, where $x_i \geq 0$ and $\sum x_i = 1$. We can form a continuous linear extension of f onto T, defining $f(x) := \sum x_i f(v_i)$. As a union of triangles, G is homeomorphic to B^2[1], so f maps B^2 to B^2, and thus f has a fixed point $x \in G$. Let $x = \sum_i x_i v_i$ for

[1]abuse of notation; this refers to the unit disk in $\mathbb{R}^2$.

some triangle v, and define $\epsilon_i \in \{\pm e_1 \pm e_2\}$, so $f(v_i) = v_i + \epsilon_i$. Then, $f(x) = x$ implies that $\sum_i x_i(v_i + \epsilon_i) = \sum_i x_i v_i$, so $\sum_i x_i \epsilon_i = 0$.

Without loss of generality, say $x_1 > 0$ and $\epsilon_1 = e_1$. Then one of the other ϵ_i must be $-e_1$, WLOG say ϵ_2. Thus, one of v_1 belongs on B_0 and the other on B_1, which is impossible as both are vertices of the same triangle. □

4 The Kneser Conjecture

Historically, the Kneser conjecture, posed by Kneser in 1955, was proven by Lovász in 1978 with a graph-theoretical proof that involved neighborhood complexes. In particular, Lovász's proof showed that the Kneser graph is not $k + 1$ colorable.

Theorem 4.1 (Kneser Conjecture). *If the collection of all n-element subsets of a set of size $2n + k$ is divided into classes such that no two sets in the same class are disjoint, at least $k + 1$ classes are needed.*

Definition 4.2 (Kneser Graph). The Kneser graph $KG(n, k)$, is the graph whose vertices correspond to the n-element subsets of a set of $2n + k$ elements. Two nodes are connected iff the two corresponding sets are disjoint.

Lovász's proof invokes the Borsuk-Ulam theorem along with a theorem by D. Gale on the even distrbution of points around a sphere. We will show a simpler version by Greene that does not use Gale's result.

Both versions of the theorem invoke the following result referred to as the Lusternik–Schnirelmann–Borsuk theorem, or LSB theorem.

Theorem 4.3 (LSB Theorem). *For any covering of S^m, the unit sphere in m dimensions, with $m + 1$ or fewer closed sets, one of the sets must contain a pair of antipodes, or points with maximal distance from each other.*

First, we require a generalization of the LSB theorem.

Lemma 4.4. *If S^m is covered with $m + 1$ sets, each of which is either open or closed, then one of the sets contains a pair of antipodes.*

Proof. We will proceed by induction on the number of closed sets t in the cover of S^m. For the pbase case of $t = 0$, we have S^m covered by open sets $U_1, \ldots, U_{m+1}$. Fix a positive number λ such that for all $x \in S^m$, the closed ball $\tilde{B}(x, \lambda)$ is contained in some U_j. It follows by compactness of S^m that there is a finite family of points $\{x_i\}$ such that the set of open balls $B(x_i, \lambda)$ cover S^m. For each j, let F_j denote the union of those $\tilde{B}(x_i, \lambda)$ contained in U_i. We have that F_j is closed, $F_j \subset U_j$, and the F_j cover S^m. Therefore, the LSB theorem implies that one of the F_j contains a pair of antipodes. Hence, one of the U_j contains a pair of antipodes as well.

Now, we proceed by strong induction. Assume that $0 < t < m + 1$ and the lemma holds for fewer than t closed sets, so we can prove it holds for t sets. Let C be a cover of S^m with $m + 1$ sets, where exactly t are closed and the rest are opne. Given a fixed closed set $F \in C$, suppose it does not contain a pair of antipodes. Hence, the diameter of this set is less than 2; define its diameter as $2 - \epsilon$ for some $\epsilon > 0$. Let U denote the open set of all points in S^m whose distance from F is less than $\epsilon/2$. Then, $(C \backslash \{F\}) \cup \{U\}$ s a cover of S^m with $m + 1$ sets, where exactly $t - 1$ are closed and the rest are open. By our inductive hypothesis, some set in thie cover must contain a pair of antipodes. But by construction, U does not contain such a pair, and thus a set in C must contain a pair of antipodes, as desired. □

Now, we are able to contain Kneser's conjecture. Define $H(a)$ as the set $\{x \in S^m \mid ax > 0\}$ as the open hemisphere centered at a, and $S(a)$ as the boundary of $H(a)$, or the set $\{x \in S^m \mid ax = 0\}$. Recall that Kneser's conjecture states that if all n-element subsets of a $2n + k$ element set are partitioned into $k + 1$ classes, at least one class must contain a pair of disjoint subsets.

Proof. Distribute $2n + k$ points on S^{k+1} such that no $k + 2$ points lie on a great k-sphere. Now, partitin the n-element subsets of these points into $k + 1$ classes $A_1, \dots, A_{k+1}$. For $i \in \{1, \dots, k + 1\}$, let U_i denote the open set of all points in S^{k+1} such that $H(a)$ contains an n-element set the class A_i. We define $F = S^{k+1} \backslash (U_1 \cup \dots \cup U_{k+1})$ as a closed set. The set F alond with the U_i are $k + 2$ sets which cover S^{k+1}, so by our lemma, one of the sets must contain a pair of antipodes $\pm a$. If this set is one of the U_i, then $H(a)$ and $H(-a)$ contian n-element subsets in the class A_i, and are clearly disjoint. Therefore, we just need to finish by showing that F cannot contain a pair of antipodes. But this is clear, as if it did, $H(a)$ and $H(-a)$ would both contian fewer than n points from the original $(2n + k)$-element set. This means that at least $k + 2$ points lie on the great k-sphere $S(a)$, which contradicts the distibution of our points.

□

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