Young Tableaux

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Abstract

This expository paper seeks to present a basic overview of Young tableaux theory — its combinatorial mechanisms, connections to building irreducible representations of the symmetric group, and applications to combinatorial objects discussed during class. This paper assumes familiarity with topics covered during class as well as basic group/representation theory concepts.

1 Overview

Young Diagrams and Tableaux occupy a central position in contemporary algebraic combinatorics. The Ferrers-board-like diagram, introduced in 1900 by Alfred Young to visually encode an integer partition and "upgraded" by filling its boxes with elements of $[n]$, became the indexing object for the representation theory of S_n. Since then, the importance of Tableaux has expanded far beyond Symmetric-Group Theory: counting the number of linear extensions of certain posets, in the underlying combinatorics of classical numbers (Eulerian, Catalan Numbers), the famous Robinson-Schensted-Knuth correspondence, and even in classifying characteristics of fundamental particles in physics.

This universality arises naturally from the study of *symmetry*: In mathematics, whenever we study structures with inherent symmetries (groups, algebras, categories, operators), it is inevitable that we study their representations. Then, in the process of decomposing these representations (which is how we study them), we can expect to encounter objects that encode stability under symmetry (the tableaux). In other words, symmetric objects can be broken down into symmetric coordinates, and Young tableaux form a particularly natural coordinate system.

In this paper, the symmetric object in question is the symmetric group S_n. Appropriately enough, we will "break it down" for study in the lens of Young tableaux theory and hint at further applications to other objects (posets). For those interested in a more comprehensive read, you may wish to take a look at Sagan: *The Symmetric Group* [4] (specifically for S_n) or Fulton: *Young Tableaux* [1] (a more comprehensive treatment of tableaux theory).

2 The Young Tableaux

2.1 Partitions and Young Diagrams

We start by introducing the basic combinatorial objects that underlie everything in this paper.

Definition 2.1. Recall that a *partition* of a positive integer n (denoted by λ) is a weakly decreasing sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k),$$

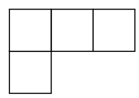
such that $\lambda_1 + \cdots + \lambda_k = n$. We write that "$\lambda \vdash n$" if λ partitions n and call $|\lambda| = n$ the *size* of λ. The numbers λ_i are called the *parts* of the partition. $\triangle$

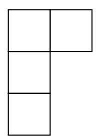
To each partition we associate a diagram:

Definition 2.2. The *Young diagram* (or *shape*) of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a left–justified array of boxes with λ_i boxes in row i, for $i = 1, \ldots, k$. Throughout this paper we use the convention where row 1 is at the top, and row numbers increase as we go down. $\triangle$

Example 2.3. *For $n = 4$ the possible partitions and their Young diagrams are*

$$(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad (1,1,1,1).$$

For instance, the partition $\lambda = (3,1) \vdash 4$ has Young diagram 

and the partition $\lambda = (2,1,1) \vdash 4$ has Young diagram  $\circ$

2.2 The Standard and Semi-standard Young Tableaux

Young diagrams become more interesting when we start filling the boxes with numbers.

Definition 2.4. Let $\lambda \vdash n$. A *Young tableau* of shape λ is a Young diagram of shape λ whose boxes are filled with positive integers. $\triangle$

Two special families of such tableaux will play a central role.

Definition 2.5. A *standard Young tableau* (abbreviated as SYT) of shape $\lambda \vdash n$ is a Young tableau of shape λ whose boxes are filled with the numbers $1, 2, \ldots, n$ in such a way that

- the entries increase strictly along each row from left to right;
- the entries increase strictly down each column from top to bottom.

$\triangle$

Definition 2.6. A *semistandard Young tableau* (SSYT) of shape λ is a Young tableau whose entries are positive integers such that

- the entries are weakly increasing along each row from left to right;
- the entries are strictly increasing down each column.

No restriction is placed on which positive integers may appear, or how many times. $\triangle$

Example 2.7. *Take* $\lambda = (3,1) \vdash 4$*. Then*

1	2	4
3		

is a standard Young tableau of shape $(3,1)$*, while*

1	1	4
3		

is semistandard but not standard, since the number 1 *is repeated.* ◦

SYTs are the basic combinatorial objects that will later encode dimensions of certain representations of the symmetric group.

2.3 A Nice Combinatorial Result

To state one of the key results, we need the notion of a *hook*.

Definition 2.8. Let λ be a partition and let (i, j) denote the box in row i and column j of the Young diagram of λ. The *hook* of the box (i, j) consists of

- the box (i, j) itself,
- all boxes in the same row to the right of (i, j),
- and all boxes in the same column below (i, j).

The *hook length* $h(i, j)$ is the number of boxes in this hook. △

Example 2.9. *Consider* $\lambda = (3,2) \vdash 5$*. If we label the boxes with their hook lengths, we get*

4	3	1
2	1	

For instance, the top-left box $(1,1)$ *has hook length* 4*, because its hook consists of* $(1,1)$ *itself, 2 boxes to its right, and one box below.* ◦

For each partition $\lambda \vdash n$, let f^λ denote the number of standard Young tableaux of shape λ. The following theorem gives a beautiful closed formula for f^λ in terms of hook lengths.

Theorem 2.10 (Hook Length Formula)**.** *Let* $\lambda \vdash n$*. Then the number of standard Young tableaux of shape* λ *is*

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)},$$

where the product runs over all boxes (i, j) *in the Young diagram of* λ*.* ★

We will present a rough outline of the proof; there are several other different combinatorial proofs in the literature (for instance, via q-analogues [2] or the RSK correspondence [3]). Instead, we emphasize the interpretation:

Remark 2.11. For each partition $\lambda \vdash n$, the quantity f^λ counts standard Young tableaux of shape λ. Later, we will see that the same number f^λ will appear as the dimension of a certain representation of the symmetric group S_n. ⋄

Proof sketch. We sketch a combinatorial proof following the probabilistic Greene–Nijenhuis–Wilf "hook walk" method [6].

Step 1: Define the hook walk. Given a cell (i, j) in the Young diagram of λ, its *hook* consists of the cell itself together with all cells to its right in the same row and all cells below it in the same column. From any cell, the hook walk moves uniformly at random to one of the cells in its hook (right or down), and the walk always eventually terminates at a *corner* (a cell of hook length 1).

Step 2: Use the hook walk to insert entries. To build a standard Young tableau, insert the labels $n, n-1, \dots, 1$ in that order. For each k, perform a hook walk on the current diagram and place k in the terminal corner cell, then remove that cell from the diagram.

Step 3: Every standard Young tableau arises with equal probability. A calculation shows that the probability the hook walk terminates at a particular corner cell is exactly the reciprocal of its hook length suitably normalized. In other words:

$$\Pr(T) = \frac{1}{n!} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}$$

for every standard Young tableau T of shape λ. When inserting $n, n-1, \dots, 1$, the hook-length factors telescope, implying that every standard Young tableau of shape λ is produced with the same probability.

Step 4: Compute the total probability. Since the algorithm produces every standard Young tableau once,

$$1 \;=\; \sum_T \Pr(T) \;=\; f^\lambda \cdot \frac{1}{n!} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}.$$

Solving for f^λ gives

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)},$$

which is the hook length formula, as desired.

□

3 Representation Theory Preliminaries

In this section, we recall the basic language of representation theory we will need for sections 4 and 5, where we explore the importance of tableaux in the representation theory of the Symmetric Group S_n. We will work over the field of complex numbers, denoted by $\mathbb{C}$.

But before getting into the details, it helps to first have a mental image of what we are trying to accomplish:

3.1 The Big Picture

So far, Young diagrams and tableaux have been purely combinatorial: we defined shapes $\lambda \vdash n$, counted the standard Young tableaux (SYT) of each shape via the hook length formula, and called this number f^λ. In the rest of the paper we attempt to explain how and why these same shapes and numbers control the representation theory of the symmetric group S_n.

Proposition 3.1. *The partitions $\lambda \vdash n$ index* all irreducible representations *of S_n. Moreover, for each λ there is an irreducible S_n–module S^λ whose dimension is exactly*

$$\dim(S^\lambda) = f^\lambda.$$

■

Sections 3, 4, and 5 are devoted to unpacking this statement.

Roughly speaking, a (complex) *representation* of a group G is a way for G to *act* on a vector space by linear transformations, so that group elements can be "seen" as matrices. Some representations split as direct sums of smaller ones; those that do not split further are called *irreducible*. Over $\mathbb{C}$, every finite-dimensional representation of a finite group is a direct sum of irreducibles (Maschke's theorem 3.9).

In our story, the symmetric group S_n acts naturally on many sets built from $[n]$, such as the set of all tableaux of a fixed shape with entries $1, \dots, n$. From such a set X we build a *permutation representation* $\mathbb{C}^X$ whose basis vectors are indexed by X and on which S_n acts by relabeling the underlying combinatorial objects. This gives us a large, concrete representation to work inside.

Here is the roadmap:

- **Section 3** reviews the minimal representation-theoretic background we need: group actions, permutation representations, irreducible representations, and Maschke's theorem.
- **Section 4** starts from the permutation representation on tabloids of a fixed shape λ and uses *Young symmetrizers* to cut out a smaller submodule S^λ, called the *Specht module*. We show that each S^λ is irreducible, and that every irreducible representation of S_n arises in this way from a unique partition $\lambda \vdash n$.
- **Section 5** connects back to the combinatorics of tableaux. Using a branching rule and Young's lattice, we prove that

  $$\dim(S^\lambda) = f^\lambda,$$

  so the hook length formula computes the dimensions of the irreducible representations of S_n.

Thus partitions and Young diagrams do not just organize tableaux; they also label all irreducible representations of S_n, with f^λ recording their dimensions.

Now, to the specifics:

3.2 Group Actions and Permutation Representations

We will promptly begin with group actions on sets, which are a more elementary version of representations.

Definition 3.2. Let G be a group and X a set. An *action* of G on X is a map

$$G \times X \to X, \qquad (g, x) \mapsto g \cdot x,$$

such that

1. $e \cdot x = x$ for all $x \in X$, where e is the identity of G;
2. $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $g_1, g_2 \in G$ and $x \in X$.

△

Definition 3.3. Recall the Symmetric Group S_n as the group containing all permutations σ of $[n]$. The symmetric group S_n acts on the set $[n]$ by

$$\sigma \cdot i = \sigma(i).$$

More generally, S_n acts on any set built from $\{1, \ldots, n\}$, for instance:

- the set of all k–element subsets of $1, \ldots, n$
- the set of all Young tableaux of a fixed shape whose entries are $1, ..., n$.

Our combinatorial objects (tableaux) will therefore naturally carry actions of S_n. △

Group actions naturally give rise to linear representations, one of which is the standard *permutation representation*.

Definition 3.4. Let G act on a finite set X. The associated *permutation representation* is the representation of G on the vector space

$$C^X = \bigoplus_{x \in X} \mathbb{C} e_x$$

with basis $\{e_x\}_{x \in X}$, where $g \in G$ acts by

$$g \cdot e_x = e_{g \cdot x}.$$

△

Remark 3.5. The direct sum

$$C^X = \bigoplus_{x \in X} \mathbb{C} e_x$$

means that every vector $v \in C^X$ can be written uniquely as a linear combination

$$v = \sum_{x \in X} a_x e_x, \qquad a_x is \in \mathbb{C},$$

where the vectors e_x form a basis indexed by the elements of X. ⋄

If X is the set of all ways to fill a fixed Young diagram with elements of $[n]$, then $\mathbb{C}X$ is a natural representation of S_n: each permutation simply relabels the entries of a tableau. In Section 4, the Specht module S^λ can be realized inside such a permutation representation.

3.3 Representations and Irreducible Pieces

We now pass from sets to vector spaces.

Definition 3.6. Let G be a group and V a complex vector space. A (linear) *representation* of G on V is a homomorphism

$$\psi : G \to \mathrm{GL}(V),$$

from G to the group of invertible linear transformations of V. We say that G "acts linearly" on V. Equivalently, specifying ψ is the same as specifying, for each $g \in G$, an invertible linear map $v \mapsto g \cdot v$ such that

$$e \cdot v = v, \qquad (g_1 g_2) \cdot v = g_1 \cdot (g_2 \cdot v).$$

Note that many people also call this representation a *G-module*. Later, when we define Specht Modules S^λ officially, we will use the words *representation* and *module* interchangeably. △

For better intuition, we provide a concrete matrix visualization of a representation of S_3:

Example 3.7. *Let $V = \mathbb{C}^3$ with standard basis e_1, e_2, e_3. For each permutation $\sigma \in S_3$ we define a linear map $\psi(\sigma)\colon V \to V$ by*

$$\psi(\sigma)(e_i) = e_{\sigma(i)}.$$

In other words, $\psi(\sigma)$ simply permutes the coordinates of a vector according to σ. With respect to the basis e_1, e_2, e_3, each $\psi(\sigma)$ is a 3×3 matrix.
For example, for the transposition $(1\ 2)$ and the 3-cycle $(1\ 2\ 3)$ we get the matrices

$$\psi((1\ 2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \psi((1\ 2\ 3)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

One checks that

$$\psi(\sigma\tau) = \psi(\sigma)\,\psi(\tau)$$

for all $\sigma, \tau \in S_3$, so $\psi\colon S_3 \to GL(V)$ is a representation in the sense of Definition 3.6. Later, when we construct Specht modules, we will often describe a representation by giving matrices for a few generators of S_n in a suitable basis; this is the same notion. ○

Inside a representation, we often want to look at invariant subspaces:

Definition 3.8. Let G be a group and V a representation of G. A subspace $W \subseteq V$ is a *subrepresentation* (or G-submodule) if $g \cdot w \in W$ for all $g \in G$ and $w \in W$. The representation V is *irreducible* if its only subrepresentations are $\{0\}$ and V itself. △

You can think of irreducible representations as the prime factors of representations: they are the ones that do not split further into smaller nontrivial invariant subspaces.

Over $\mathbb{C}$, finite-dimensional representations of finite groups always decompose into irreducibles:

Theorem 3.9 (Maschke's Theorem)**.** *Let G be a finite group and let V be a complex representation of G. Then V decomposes as a direct sum of irreducible subrepresentations:*

$$V \cong W_1 \oplus \cdots \oplus W_r,$$

where each W_i is irreducible. ★

A proof will not be given for the sake of pacing (one may check out various published literature; e.g. [5])

We will use Maschke's Theorem in the background: once we have constructed a supply of irreducible S_n–modules S^λ (the Specht Modules), every finite-dimensional representation of S_n can be built by taking direct sums of them, according to the theorem.

3.4 Conjugacy Classes of S_n and Partitions

Just before we actually construct our *irreps*, we offer a few key details of the symmetric group and attempt to highlight some of the intuition behind why the Young diagrams/tableaux appear in the representations of S_n:

Definition 3.10. Recall that two elements $g, h \in G$ are *conjugate* if there exists $x \in G$ such that $h = xgx^{-1}$. Note that the equivalence classes for this relation are the *conjugacy classes* of G. △

In group theory, the idea of *conjugacy classes* allows mathematicians to organize the group. As it turns out, elements that are *conjugate* behave the same way inside *every* representation; so, from the point of view of representations, they are indistinguishable. For S_n, these classes are indexed by partitions of n via cycle type:

Theorem 3.11. *In S_n, two permutations are conjugate if and only if they have the same* ***cycle type****. Equivalently, the conjugacy classes of S_n are in bijection with the partitions of n.* ★

This is not hard to see:

Proof. Consider the act of conjugation in S_n simply as a relabeling of the elements when the permutation is written in cycle notation. If

$$\pi = (a_1 ... a_k)(b_1 ... b_l) ... \in S_n$$

and

$$\sigma \in S_n$$

sends x to x', then

$$\sigma\pi\sigma^{-}1 = (a'_1 ... a'_k)(b'_1 ... b'_l) ....$$

Thus, the conjugacy classes of S_n are characterized by the cycle types. □

Moreover, since the conjugacy classes are characterized by cycle types, they must correspond to partitions of n (i.e. the *Young diagrams* of size n).

Finally, recall from representation theory that:

Theorem 3.12. *the number of irreducible representations of a finite group is equal to the number of its conjugacy classes.* ★

From this, it becomes quickly obvious why the Young diagrams/tableaux appear in the representation theory of S_n. There are exactly $p(n)$ conjugacy classes of S_n: each of them corresponds one-to-one with a partition $\lambda \vdash n$ of n. Hence, there must be underlying symmetries encoded by the Young tableaux that allow for this correspondence. Preserving that invariance and using it to build irreducible representations is going to be our goal for the next section.

4 Constructing Irreducible Representations of S_n

Finally, we use Young tableaux to build concrete models for irreducible representations of S_n. The main tools we will use are *Young Symmetrizers* and the associated *Specht Modules* S^λ.

The starting point is the permutation representation (3.4): fix a partition $\lambda \vdash n$, and let X be the set of all tableaux of shape λ whose boxes are filled with the numbers $1, ..., n$. Then, from 3.2 and 3.3: S_n acts on X by permuting the labels, and hence on the vector space $\mathbb{C}X$. This representation contains many invariant subspaces, and is thus usually far from irreducible.

That's where we get the idea of *symmetrizers*: we want to "filter out" from $\mathbb{C}X$, a distinguished irreducible piece with a specific symmetry pattern determined by λ: symmetric within each row, and alternating ("antisymmetric") within each column. We implement this using elements of the group algebra $\mathbb{C}[S_n]$ called **Young Symmetrizers**. One can think of applying a Young symmetrizer to $\mathbb{C}[S_n]$ like a projection onto our corresponding Specht module S^λ.

*An explicit construction example (S_3) will be shown at end of this section for reference.

Remark 4.1. The (complex) **group algebra** of S_n (denoted as $\mathbb{C}[S_n]$ for S_n specifically), concretely speaking, is the vector space whose basis elements are the permutations in S_n, where we allow formal linear combinations

$$\sum_{\sigma \in S_n} a_\sigma \, \sigma, \qquad a_\sigma \in \mathbb{C},$$

and we multiply them by extending the group multiplication in S_n linearly. Thus, an element of $\mathbb{C}[S_n]$ is just a "blend" of permutations, and such elements act on a representation by applying each permutation and adding the results. The Young symmetrizers we use are particular elements of $\mathbb{C}[S_n]$ with carefully chosen coefficients. Their construction is given below: $\diamond$

4.1 Row and Column Groups → Young Symmetrizers

Definition 4.2. Given a tableau T of shape $\lambda \vdash n$, the *row group* R_T is the subgroup of S_n consisting of all permutations that preserve each row of T as a set. Similarly, we analogously define the *column group* C_T is the subgroup of S_n consisting of all permutations that preserve each column of T as a set. $\triangle$

Example 4.3. *Let* $\lambda = (2, 1) \vdash 3$ *and consider the tableaux*

<table><tr><td>1</td><td>2</td></tr><tr><td>3</td><td></td></tr></table>

The row group R_T *consists of the identity and the transposition* $(1\,2)$*, which swaps the two entries in the first row. The column group* C_T *consists of the identity and the transposition* $(1\,3)$*, which swaps the two entries in the first column.* $\circ$

We now pass from subgroups to elements in the group algebra $\mathbb{C}[S_n]$.

Definition 4.4. The *row symmetrizer* and *column antisymmetrizer* associated to T are the elements

$$a_T = \sum_{\sigma \in R_T} \sigma, \qquad b_T = \sum_{\tau \in C_T} \operatorname{sgn}(\tau)\, \tau$$

in the group algebra $\mathbb{C}[S_n]$. Their product

$$c_T = a_T b_T \in \mathbb{C}[S_n]$$

is exactly the *Young Symmetrizer* associated to the tableau T. $\triangle$

Remark 4.5. In the column antisymmetrizer

$$b_T = \sum_{\tau \in C_T} \operatorname{sgn}(\tau)\, \tau,$$

the factor $sgn(\tau)$ denotes the sign of the permutation τ: it is $+1$ for even permutations and -1 for odd permutations. Including this sign ensures that b_T is alternating with respect to permutations of the entries within each column of the tableau T. In contrast, the row symmetrizer a_T produces a symmetric sum over permutations in each row. ◇

Remark 4.6. The adjectives "symmetrizer" and "antisymmetrizer" reflect the following heuristic: a_T averages over permutations in the rows (forcing symmetry within each row), while b_T takes an alternating sum over permutations in the columns (forcing antisymmetry within each column). The Young symmetrizer c_T imposes both constraints at once. ◇

Example 4.7. *We explicitly construct the Young Symmetrizer of*

$$T := \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

from example 4.3, for which its row and column groups are

$$R_T = \{e, (12)\}, \qquad C_T = \{e, (13)\} \subset S_3,$$

respectively.
The associated row symmetrizer and column antisymmetrizer are

$$a_T = \sum_{\sigma \in R_T} \sigma = e + (12), \qquad b_T = \sum_{\tau \in C_T} \operatorname{sgn}(\tau)\, \tau = e - (13)$$

in the group algebra $\mathbb{C}[S_3]$. *Their product*

$$\begin{aligned} c_T = a_T b_T &= (e + (12))(e - (13)) \\ &= e - (13) + (12) - (12)(13) \end{aligned}$$

is the Young symmetrizer associated to T. *Using the convention that products of permutations are composed from right to left, we have* $(12)(13) = (132)$, *so we can also write*

$$c_T = e - (13) + (12) - (132) \in \mathbb{C}[S_3].$$

To see concretely how c_T *imposes both symmetry conditions, let* V *be any* S_3*–representation and* $v \in V$.
First apply the column antisymmetrizer:

$$b_T v = v - (13) \cdot v.$$

Acting by the column permutation (13) *gives*

$$(13) \cdot (b_T v) = (13) \cdot v - (13)^2 \cdot v = (13) \cdot v - v = -\big(v - (13) \cdot v\big) = -\, b_T v,$$

so $b_T v$ *changes sign when we swap the entries in the first column. In this sense,* b_T *enforces "alternating under the column group".*
Now apply the row symmetrizer to $b_T v$*:*

$$w := a_T(b_T v) = (e + (12))(b_T v) = b_T v + (12) \cdot (b_T v).$$

Acting by the row permutation (12), *we obtain*

$$(12) \cdot w = (12) \cdot (b_T v) + (12)^2 \cdot (b_T v) = (12) \cdot (b_T v) + b_T v = w.$$

Thus w *is fixed by* (12), *so it is symmetric under permutations in the row group.*
Since $c_T v = a_T b_T v = w$, *the Young symmetrizer* c_T *produces vectors that are simultaneously symmetric along the rows of* T *and alternating along its columns, as desired.* ○

Using the symmetrizers, we can now build the irreducible representations of S_n:

4.2 Specht Modules

Definition 4.8. Let $\lambda \vdash n$ and let T be any tableaux of shape λ. The **Specht module** of shape λ is the $\mathbb{C}[S_n]$–submodule

$$S^\lambda = \mathbb{C}[S_n]\, c_T \subseteq \mathbb{C}[S_n]$$

generated by the Young symmetrizer c_T. More concretely speaking, S^λ is defined as the subspace of $\mathbb{C}[S_n]$ spanned by the vectors σc_T such that each of its vectors is a linear combination of the form

$$\sum_{\sigma \in S_n} a_\sigma(\sigma c_T), \quad a_\sigma \in \mathbb{C}.$$

Of course, these vectors, in general, are not linearly independent, and so its dimension is not $n!$. △

One can think of the Specht Module as the sub-representation of the regular representation of S_n where we have enforced symmetry along the rows of T (via a_T) and antisymmetry along the columns (via b_T). Naturally, it makes sense that any two Tableaux with the same shape λ would encode the same symmetry patterns, hence making each of their respective Specht Modules isomorphic:

Theorem 4.9. *If T and T' are tableaux of the same shape λ, then the S_n–modules $\mathbb{C}[S_n]c_T$ and $\mathbb{C}[S_n]c_{T'}$ are isomorphic. In particular, the Specht module S^λ is well defined up to isomorphism.* ★

Proof. There is a quick proof for this result:
Let T and T' be tableaux of the same shape λ. There exists a permutation $\sigma \in S_n$ that sends the filling of T to that of T'. Then:

$$\sigma R_T \sigma^{-1} = R_{T'} \qquad \text{and} \qquad \sigma C_T \sigma^{-1} = C_{T'},$$

so conjugation by σ carries the row and column groups of T to those of T'.
Passing to the group algebra, we obtain

$$\sigma a_T \sigma^{-1} = a_{T'} \qquad \text{and} \qquad \sigma b_T \sigma^{-1} = b_{T'},$$

and hence

$$\sigma c_T \sigma^{-1} = \sigma a_T b_T \sigma^{-1} = (\sigma a_T \sigma^{-1})(\sigma b_T \sigma^{-1}) = a_{T'} b_{T'} = c_{T'}.$$

Now, define a map

$$\phi : \mathbb{C}[S_n]c_T \longrightarrow \mathbb{C}[S_n]c_{T'}, \qquad \phi(gc_T) = g\sigma^{-1}c_{T'}.$$

This map is well defined and $\mathbb{C}$–linear. Moreover, notice that it is an S_n–module homomorphism: for any $h \in S_n$, we have

$$\phi\big(h \cdot (gc_T)\big) = \phi\big((hg)c_T\big) = (hg)\sigma^{-1}c_{T'} = h \cdot (g\sigma^{-1}c_{T'}) = h \cdot \phi(gc_T).$$

Finally, an inverse map is given by

$$\psi : \mathbb{C}[S_n]c_{T'} \longrightarrow \mathbb{C}[S_n]c_T, \qquad \psi(gc_{T'}) = g\sigma c_T,$$

so ϕ is also bijective and hence an isomorphism of S_n–modules. Therefore $\mathbb{C}[S_n]c_T \cong \mathbb{C}[S_n]c_{T'}$ whenever T and T' have the same shape, and in particular the Specht module S^λ is well defined up to isomorphism, as desired. □

Thus for each partition $\lambda \vdash n$, we obtain a well-defined representation S^λ of S_n.

To show that the Specht modules are indeed irreducible over $\mathbb{C}[S_n]$, we may use *tabloids* and *polytabloids*:

Definition 4.10 (Tabloids)**.** Given two tableau of shape λ (i.e. a "λ-tableau") are called *row-equivalent* if each row contains the same set of entries (possibly in a different order). A λ*-tabloid* is a row-equivalence class of λ-tableaux. We write $\{T\}$ for the tabloid represented by a tableau T. The group S_n acts on tabloids by permuting the entries:

$$\sigma\{T\} = \{\sigma T\}, \quad \sigma \in S_n.$$

△

Definition 4.11 (Polytabloids)**.** The polytabloid associated to T is the vector:

$$e_T = c_T\{T\}$$

in the permutation representation. △

In other words: e_T is the signed linear combination of tabloids obtained by symmetrizing along rows and antisymmetrizing along columns. Note that the Specht module S^λ is defined to be the span of all polytabloids e_T of shape λ.

Fix a partition $\lambda \vdash n$. Let X be the set of all λ–tabloids. By Definition 3.4, the permutation representation of S_n on X is the vector space

$$M^\lambda := \mathbb{C}^X,$$

which we call the *permutation module of shape* λ.

Now, the irreducibility of S^λ:

Theorem 4.12 (Specht modules are irreducible)**.** *For each partition $\lambda \vdash n$, the Specht module S^λ is an irreducible $\mathbb{C}[S_n]$–module. For fixed n, the modules S^λ with $\lambda \vdash n$ form a complete set of pairwise non-isomorphic irreducible representations of S_n.* ★

Idea of the proof. We briefly sketch why this is plausible and refer to *The Symmetric Group* [4, §2.7] for details.

First, one shows that S^λ is generated by any single standard polytabloid e_T: the S_n–orbit of e_T contains all polytabloids e_U with U standard, and these span S^λ. Thus S^λ is a cyclic $\mathbb{C}[S_n]$–module.

Next, one proves that there are certain linear relations among polytabloids, called *Garnir relations*, which allow any nonstandard polytabloid to be rewritten as a linear combination of polytabloids coming from "more standard-looking" tableaux. Using these relations, any nonzero vector in S^λ can be *straightened* until a single standard polytabloid appears with nonzero coefficient. In particular, every nonzero submodule of S^λ contains some standard polytabloid.

Combining these two facts shows that any nonzero submodule of S^λ must already contain a generator of S^λ, and hence equals S^λ itself. Therefore S^λ is irreducible. The statement that the various S^λ exhaust all irreducible representations of S_n is proved using character theory and will not be discussed here. □

4.3 The "Main Structure Theorem"

From this well-defined representation using Specht Modules, we reach (part of) the claim we set up all the way back in 3.1:

Theorem 4.13. *For each partition $\lambda \vdash n$, the Specht module S^λ is an irreducible representation of S_n. Moreover, every irreducible complex representation of S_n is isomorphic to S^λ for a unique partition λ of n. If $\lambda \neq \mu$, then S^λ and S^μ are non-isomorphic.* ★

Taken together with Maschke's Theorem, this shows that the Specht modules provide a complete set of building blocks for representation theory of S_n: any finite-dimensional representation of S_n decomposes as a direct sum of Specht modules.

As promised, we will explicitly construct the irreps of S_3 as an example:

Example 4.14. *For $n = 3$ there are three partitions*

$$(3), \qquad (2,1), \qquad (1,1,1),$$

so by Theorem 4.13 the three Specht modules $S^{(3)}, S^{(2,1)}, S^{(1,1,1)}$ give all irreducible representations of S_3. We now describe them explicitly.

(1) The Specht module $S_{(3)}$ (the trivial representation). *Take the young tableau:*

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$$

of shape (3). Here the row group is all of S_3 and the column group is trivial:

$$R_T = S_3, \qquad C_T = \{e\}.$$

Thus the row symmetrizer and column antisymmetrizer are

$$a_T = \sum_{\sigma \in S_3} \sigma, \qquad b_T = e,$$

so the Young symmetrizer is $c_T = a_T b_T = a_T$. The Specht module

$$S^{(3)} = \mathbb{C}[S_3]c_T$$

is one–dimensional, spanned by c_T itself. For any $\tau \in S_3$ we have

$$\tau \cdot c_T = \tau \sum_{\sigma \in S_3} \sigma = \sum_{\sigma \in S_3} \tau\sigma = \sum_{\sigma' \in S_3} \sigma' = c_T,$$

*after reindexing $\sigma' = \tau\sigma$. Thus every group element acts as the identity on $S^{(3)}$, so this Specht module is exactly the **trivial representation** of S_3.*

(2) The Specht module $S^{(1,1,1)}$ (the sign representation). *Now, take the vertical tableau*

$$T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

of shape $(1,1,1)$. In this case the row group is trivial and the column group is all of S_3:

$$R_T = \{e\}, \qquad C_T = S_3.$$

Hence

$$a_T = e, \qquad b_T = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, \sigma,$$

so $c_T = a_T b_T = b_T$. Again $S^{(1,1,1)} = \mathbb{C}[S_3]c_T$ is one–dimensional, spanned by c_T, but now for $\tau \in S_3$ we have

$$\begin{aligned} \tau \cdot c_T &= \tau \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, \sigma = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma)\, \tau\sigma \\ &= \sum_{\sigma' \in S_3} \operatorname{sgn}(\tau^{-1}\sigma')\, \sigma' = \operatorname{sgn}(\tau) \sum_{\sigma' \in S_3} \operatorname{sgn}(\sigma')\, \sigma' = \operatorname{sgn}(\tau)\, c_T. \end{aligned}$$

*Thus each permutation acts by the scalar $\operatorname{sgn}(\tau)$, so $S^{(1,1,1)}$ is precisely the **sign representation** of S_3.*

(3) The Specht module $S^{(2,1)}$ (the standard representation). *Finally, consider the tableau from Example 4.3,*

$$T \; = \; \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$

of shape $(2,1)$. We have already computed in Example 4.7 that

$$R_T = \{e, (12)\}, \qquad C_T = \{e, (13)\},$$

and the corresponding Young symmetrizer is

$$c_T = e - (13) + (12) - (132) \in \mathbb{C}[S_3].$$

The Specht module $S^{(2,1)} = \mathbb{C}[S_3]c_T$ is a submodule of the regular representation of S_3. As for its dimension, we will simply state that it is 2-dimensional. The reason behind this will be given in the next section (although you may already have guessed that it has to due with the hook length of T; which in our case equals 2).
For a concrete description, set

$$v_1 = c_T, \qquad v_2 = (23)\, c_T.$$

One can check that v_1 and v_2 are linearly independent and span $S^{(2,1)}$, so they form a basis. In this basis, the generators (12) and (23) of S_3 act as

$$(12) \cdot v_1 = v_1, \qquad (12) \cdot v_2 = -v_1 - v_2,$$

$$(23) \cdot v_1 = v_2, \qquad (23) \cdot v_2 = v_1.$$

Thus the matrices of these elements are

$$\psi_{(2,1)}(12) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad \psi_{(2,1)}(23) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

*which is the usual 2–dimensional **standard representation** of S_3, as desired.*

Altogether, the three Specht modules $S^{(3)}$, $S^{(2,1)}$, and $S^{(1,1,1)}$ have dimensions 1, 2, and 1, respectively, and are pairwise non-isomorphic. Their dimensions satisfy

$$1^2 + 2^2 + 1^2 = 6 = |S_3|,$$

so, in accordance with theorem 4.13, they indeed account for all irreducible representations of S_3! ◦

In the next section we will connect these modules back to the combinatorics of standard Young tableaux. In particular, we will see that

$$\dim(S^{\lambda}) = f^{\lambda},$$

as indicated during our example.

The fact that the Hook Length Formula from theorem 2.10 gives an explicit formula for the dimensions of all irreducible representations of the symmetric group is certainly one of the most beautiful results from the intersection between abstract algebra and combinatorics.

5 Dimension Formula via Hook Lengths

5.1 A brief reminder: Restriction, Induction, and the idea behind "branching"

Before discussing the branching rule for Specht modules (which is essential to proving the explicit formula for $\dim(S^\lambda)$), we pause to recall two basic operations on representations of symmetric groups. These ideas will clarify the notation $\mathrm{Res}^{S_n}_{S_{n-1}}$ and explain why removing a box from λ is the correct combinatorial move when passing from S_n to S_{n-1}.

Definition 5.1 (Restriction)**.** If V is a representation of S_n, we may view S_{n-1} as the subgroup of permutations that fix n. The *restriction*

$$\mathrm{Res}^{S_n}_{S_{n-1}} V$$

is the same vector space V, but we only allow the elements of S_{n-1} to act. Conceptually, we "forget that n exists" and look only at how the remaining letters $\{1, \dots, n-1\}$ are permuted. △

For some intuition: A Specht module S^λ is built using Young symmetrizers, which depend on how the numbers $1, 2, \dots, n$ appear inside the diagram λ. If we restrict to S_{n-1}, the letter n no longer moves, so the only relevant information is how the remaining boxes of λ (those whose entries are $1, \dots, n-1$ in a tableau) transform. This means the shape λ should "lose" the box containing n.

Definition 5.2 (Induction)**.** The opposite operation is *induction*: given a representation W of S_{n-1}, one forms

$$\mathrm{Ind}^{S_n}_{S_{n-1}} W,$$

which is a canonical way to let all of S_n act. We do not need the details here, but the key idea is that induction corresponds to *adding* the letter n in all possible ways. This process is closely related to the edges in Young's lattice obtained by adding a box. △

For some more intuition: Consider S_3 and its subgroup $S_2 = \langle e, (1\ 2)\rangle$. If T is a tableau of shape $(2, 1)$, removing the box containing 3 always produces a tableau of shape (2) or $(1, 1)$. These are exactly the shapes that appear in the restriction of $S^{(2,1)}$:

$$\mathrm{Res}^{S_3}_{S_2} S^{(2,1)} \cong S^{(2)} \oplus S^{(1,1)}.$$

Thus, the representation–theoretic behavior mirrors the combinatorics of removing a corner box.

This perspective is the guiding idea behind the branching rule: when we restrict from S_n to S_{n-1}, the box containing n must be removed, and every possible removal contributes one summand. The next section makes this precise.

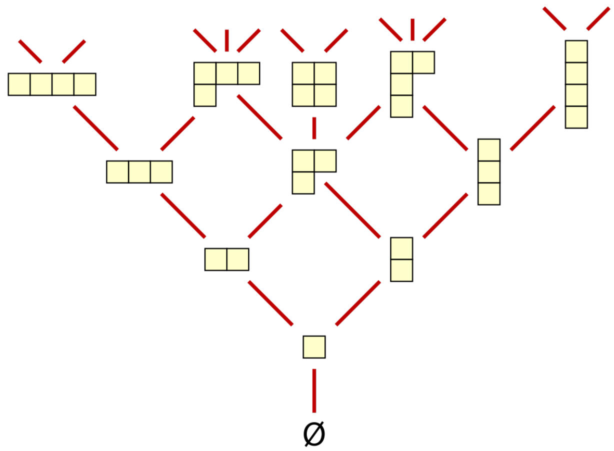
5.2 Young's Lattice and the Branching Rule

In the previous subsection, we saw that restricting a representation from $S_n \to S_{n-1}$ should be thought of as "forgetting" the letter n (for Specht modules, this process will be reflected combinatorially by removing a single box from the Young diagram). Here, we introduce *Young's lattice*: the graph that records all such box-removals. It is exactly the combinatorial shadow of the branching rule.

Definition 5.3 (Young's Lattice)**.** A box (i, j) of a Young diagram λ is called a *corner* if there is no box directly to its right and no box directly below it. If μ is obtained from λ by removing a single corner box, we write $\mu \to \lambda$. △

The relation $\mu \to \lambda$ turns the set of all partitions into a graph called *Young's lattice*: we draw an edge from μ to λ whenever $\mu \to \lambda$. Paths of length n from the empty partition to a partition $\lambda \vdash n$ correspond to sequences of diagrams obtained by adding one box at a time.

Example 5.4. *Below is the illustration of such a lattice. Here, we have the diagrams of each partition of S_n (from the trivial S_0 to S_4) which corresponds to each S_n's irreducible representations:*



As we move up the lattice, each horizontal "row" consists of all partitions of a fixed integer n, and hence of all irreducible representations of S_n:

- *Level $n = 0$: The bottom $\emptyset$ is the unique representation of 0. It corresponds to the trivial representation of S_0.*
- *Level $n = 1$: The diagram (1) is the only partition of 1, so S_1 also has just one irreducible representation.*
- *Level $n = 2$: The two diagrams (2) and (1,1) give the two irreducible representations of S_2: the trivial representation and the sign representation.*
- *Level $n = 3$: The three diagrams (3), (2,1), and (1,1,1) give the three irreducible representations of S_3: the trivial representation, the 2-dimensional standard representation, and the sign representation (this was shown in Example 4.14).*
- *Level $n = 4$: The five diagrams (4), (3,1), (2,2), (2,1,1), and (1,1,1,1) give the five irreducible representations of S_4. Later, the hook length formula will compute each of their respective dimensions.*

∘

In general, the n-th level of Young's lattice contains all partitions of n, so walking along that level is the same as listing all irreducible representations of S_n. The branching rule presented below will tell us how these representations are connected by the edges of the lattice.

Theorem 5.5 (Branching Rule for S_n)**.** *Let $\lambda \vdash n$. When we restrict the Specht module S^λ from S_n to S_{n-1}, it decomposes as*

$$\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda \cong \bigoplus_{\mu \to \lambda} S^\mu,$$

where the sum runs over all partitions μ obtained from λ by removing a single corner box.
Furthermore, taking dimensions of the S_μ's gives a recursion for $\dim(S^\lambda)$:

$$\dim(S^\lambda) = \sum_{\mu \to \lambda} \dim(S^\mu). \tag{1}$$

★

Proof sketch. We will not prove this theorem here, but we briefly sketch the combinatorial idea. One can notice that the Specht module S^λ in a way that admits a natural basis $\{v_T\}$ indexed by standard Young tableaux T of shape λ. The precise construction of v_T will not matter for us; we only use the indexing by tableaux.

Now restrict from S_n to S_{n-1}. Elements of S_{n-1} only permute the numbers $\{1, \dots, n-1\}$ and leave n fixed. A key observation is that in any standard Young tableau of shape λ, the entry n must sit in a corner box of λ: if there were a box to its right or below, that box would have to contain a larger number, which is impossible.

Thus every basis vector v_T singles out a corner box of λ, namely the one containing n. For each corner c of λ, consider the subspace

$$V_c = \mathrm{span}\{\, v_T : \text{in } T,\ n \text{ lies in the box } c\,\}.$$

Because permutations in S_{n-1} never move the letter n, they cannot move n out of its corner, so each V_c is stable under the action of S_{n-1}.

If we erase the box containing n from any tableau contributing to V_c, we obtain a standard Young tableau of some smaller shape μ, where μ is obtained from λ by removing the corner box c. Conversely, any standard tableau of shape μ can be extended uniquely to a standard tableau of shape λ by inserting n into the corner c. Thus the tableaux indexing V_c are in natural bijection with standard Young tableaux of shape μ, and from the point of view of S_{n-1} the subspace V_c behaves like a Specht module of shape μ.

Different corners c give different shapes μ, and together the subspaces V_c span all of S^λ. This leads to the decomposition

$$\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda \cong \bigoplus_{\mu \to \lambda} S^\mu$$

stated in the branching rule. □

5.3 The Same Recursion for Standard Young Tableaux

Now look at the numbers f^λ. Fix a partition $\lambda \vdash n$ and consider a standard Young tableau T of shape λ. If we erase the box containing n, we obtain a standard Young tableau T' of shape μ, where μ is obtained from λ by removing that corner box. This gives a map

$$\{\text{SYT of shape } \lambda\} \longrightarrow \bigcup_{\mu \to \lambda} \{\text{SYT of shape } \mu\}.$$

Conversely, starting from a standard Young tableau T' of shape $\mu \to \lambda$, we can insert n into the unique box that must be added to μ to obtain λ; this clearly produces a standard Young tableau of shape λ. Thus, we have a **bijection** between:

- SYT of shape λ, and
- pairs (μ, T') where $\mu \to \lambda$ and T' is an SYT of shape μ.

Counting both sides gives the recursion

$$f^\lambda = \sum_{\mu \to \lambda} f^\mu. \tag{2}$$

Comparing (1) and (2), we see that the dimensions of Specht modules and the numbers of standard Young tableaux satisfy the same branching rule.

5.4 Equality of Dimensions and Tableaux Counts

Finally, we can prove the promised equality.

Theorem 5.6. *For each partition $\lambda \vdash n$, the dimension of the Specht module S^λ equals the number of standard Young tableaux of shape λ:*

$$\dim(S^\lambda) = f^\lambda.$$

★

Proof sketch. We argue by induction on n.
Base case: For $n = 1$ there is only one partition $\lambda = (1)$; the Specht module $S^{(1)}$ is the 1–dimensional trivial representation, and there is also exactly one SYT of shape (1), so $\dim(S^{(1)}) = f^{(1)} = 1$.
Inductive step: Assume the statement holds for all partitions of $n-1$. Let $\lambda \vdash n$. Using the branching rule (Theorem 5.5) and taking dimensions, we get

$$\dim(S^\lambda) = \sum_{\mu \to \lambda} \dim(S^\mu).$$

By the inductive hypothesis, for each $\mu \to \lambda$ we have $\dim(S^\mu) = f^\mu$, so

$$\dim(S^\lambda) = \sum_{\mu \to \lambda} f^\mu.$$

But the combinatorial argument above shows that $f^\lambda = \sum_{\mu \to \lambda} f^\mu$, giving $\dim(S^\lambda) = f^\lambda$, as desired. □

Combining this with the Hook Length Formula now yields an explicit formula for the dimensions of all irreducible representations of S_n:

Corollary 5.7 (The explicit formula for $\dim(S^\lambda)$)**.** *For each partition $\lambda \vdash n$,*

$$\dim(S^\lambda) = f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}.$$

†

As promised, we provide an example – computing the dimensions of all Specht modules of S_4:

Example 5.8. *The partitions of 4 are*

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,1,1,1).$$

For each $\lambda \vdash 4$ we compute $\dim(S^\lambda) = f^\lambda$ using the hook length formula.

- $\boldsymbol{\lambda = (4)}$**.** *The Young diagram is a single row of four boxes. The hook lengths are 4, 3, 2, 1, so*

$$f^{(4)} = \frac{4!}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{24}{24} = 1.$$

  Thus $S^{(4)}$ is 1-dimensional (the trivial representation).

- $\boldsymbol{\lambda = (3,1)}$**.** *The hook lengths are*

4	2	1
1		

  so their product is $4 \cdot 2 \cdot 1 \cdot 1 = 8$, and

$$f^{(3,1)} = \frac{4!}{8} = \frac{24}{8} = 3.$$

  Thus $S^{(3,1)}$ is 3-dimensional.

- $\boldsymbol{\lambda = (2,2)}$**.** *The hook lengths are*

3	2
2	1

  with product $3 \cdot 2 \cdot 2 \cdot 1 = 12$, *hence*

  $$f^{(2,2)} = \frac{4!}{12} = \frac{24}{12} = 2.$$

  Thus $S^{(2,2)}$ *is* 2*-dimensional.*

- $\boldsymbol{\lambda = (2,1,1)}$**.** *The hook lengths are*

4	1
2	
1	

  so the product is $4 \cdot 1 \cdot 2 \cdot 1 = 8$, *and*

  $$f^{(2,1,1)} = \frac{4!}{8} = \frac{24}{8} = 3.$$

  Thus $S^{(2,1,1)}$ *is* 3*-dimensional.*

- $\boldsymbol{\lambda = (1,1,1,1)}$**.** *The hook lengths are:*

4	3	2	1

  Thus

  $$f^{(1,1,1,1)} = \frac{4!}{4 \cdot 3 \cdot 2 \cdot 1} = 1,$$

  and $S^{(1,1,1,1)}$ *is* 1*-dimensional (the sign representation).*

Summarizing, the dimensions of the Specht modules of S_4 *are:*

$$\dim(S^{(4)}) = 1, \quad \dim(S^{(3,1)}) = 3, \quad \dim(S^{(2,2)}) = 2, \quad \dim(S^{(2,1,1)}) = 3, \quad \dim(S^{(1,1,1,1)}) = 1.$$

One checks that

$$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 24 = 4!,$$

as expected from the general identity $\sum_{\lambda \vdash n} (\dim(S^\lambda))^2 = n! = |S_n|$. $\circ$

6 Young Tableaux and Linear Extensions of Posets

Up to this point, Young diagrams and Young tableaux have appeared mainly in the context of representation theory: they index the Specht modules and hence the irreducible representations of S_n (see the S_3 example above). In this section we switch gears and look at a purely combinatorial avatar of the same objects: *linear extensions* of partially ordered sets (posets). Later we briefly hint at how generating functions associated to these posets lead naturally to P-partitions and Schur functions.

6.1 Posets and Linear Extensions

We recall some basic definitions from class and add some extra insights.

Definition 6.1. A *partially ordered set* (abbreviated as *poset*) is a pair $(P, \leq)$ where P is a set and $\leq$ is a binary relation on P that is **reflexive**, **antisymmetric**, and **transitive**. If $x \leq y$ and $x \neq y$ we write $x < y$. If neither $x \leq y$ nor $y \leq x$ holds, we say x and y are *incomparable.* △

Definition 6.2. Let $(P, \leq)$ be a finite poset with $|P| = n$. A *linear extension* of P is a bijection

$$\phi : P \longrightarrow \{1, 2, \ldots, n\}$$

such that whenever $x \leq y$ in P we have $\phi(x) \leq \phi(y)$. Equivalently, a linear extension is a total order on P that is compatible with the partial order. We denote by $e(P)$ the number of linear extensions of P. △

Speaking generally, a linear extension is the same thing as a permutation

$$x_1, x_2, \ldots, x_n$$

of the elements of P such that $x_i \leq x_j$ in P implies $i \leq j$. This viewpoint will match perfectly with the standard Young tableaux.

Example 6.3. *Let P be the poset with elements $\{a, b, c, d\}$ and relations*

$$a < b, \qquad a < c, \qquad b < d, \qquad c < d,$$

and no other comparabilities. One can picture P as a diamond-shaped Hasse diagram: a at the bottom, b and c in the middle, and d at the top. A linear extension of P is an ordering of $\{a, b, c, d\}$ that respects all these inequalities. For instance,

$$a < b < c < d \quad \text{and} \quad a < c < b < d$$

are linear extensions, but $b < a < c < d$ is not (it violates $a < b$).
A short case check shows that

$$e(P) = 2,$$

since the middle two elements b and c can be swapped but must both appear after a and before d. ◦

In general, computing $e(P)$ for an arbitrary poset P is difficult. For the special family of posets coming from Young diagrams, however, the answer is famously beautiful and (hopefully) predictable:

6.2 Ferrers posets and standard Young tableaux

Fix a partition $\lambda \vdash n$ and its Young diagram. We can regard the cells of this diagram as forming a poset in a very natural way.

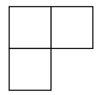
Definition 6.4. Let λ be a partition and consider its Young diagram as a left-justified array of boxes. Index the boxes by coordinates (i,j) where i is the row and j is the column. The *Ferrers poset* P_λ is the poset whose elements are the boxes of λ and whose partial order is given by

$$(i,j) \leq (i',j') \quad \iff \quad i \leq i' \text{ and } j \leq j'.$$

In other words, a box must lie *weakly northwest* of another box in order to be less than or equal to it. △

Note that if two boxes lie in the same row, then the left one $\leq$ the right one, and if they lie in the same column, then the top one $\leq$ the bottom one.

Example 6.5. *Take $\lambda = (2,1)$, whose Young diagram is*



with boxes which we label as

$$(1,1),\ (1,2) \text{ on the first row, and } (2,1) \text{ on the second row.}$$

Then P_λ has relations

$$(1,1) \leq (1,2), \qquad (1,1) \leq (2,1),$$

and no other comparabilities (since $(1,2)$ and $(2,1)$ are incomparable). Up to relabeling of elements, this is exactly the "diamond" poset from the previous example. ○

The key observation is that standard Young tableaux of shape λ are **exactly the same thing as linear extensions of P_λ.**

Proposition 6.6. *Let $\lambda \vdash n$ and let P_λ be its Ferrers poset. Then there is a natural bijection*

$$\{\text{standard Young tableaux of shape } \lambda\} \longleftrightarrow \{\text{linear extensions of } P_\lambda\}.$$

In particular, the number of standard Young tableaux of shape λ equals $e(P_\lambda)$. ■

Proof. We describe inverse constructions to prove bijectiveness.

• *From a standard Young tableau to a linear extension.* Let T be a standard Young tableau of shape λ. By definition, T fills the boxes of λ with the numbers $1, 2, \ldots, n$ so that entries are strictly increasing along rows (left to right) and along columns (top to bottom).

Read off the boxes of λ in the order in which their labels appear: first the box containing 1, then the box containing 2, and so on up to n. This gives a total order on the boxes:

$$b_1, b_2, \ldots, b_n,$$

where b_k is the unique box containing the label k in T.

If $(i,j) \leq (i',j')$ in P_λ, then the box (i,j) lies weakly northwest of (i',j'), so (i,j) is either to the left of or above (i',j'). The row and column strictness conditions in T imply that the number in (i,j) is smaller than the number in (i',j'). Therefore, in the ordering $b_1, \ldots, b_n$, the box (i,j) appears before (i',j').Thus this total order is a linear extension of P_λ.

Alternatively, we may define a map

$$\phi : P_\lambda \to \{1, 2, \ldots, n\}, \qquad \phi(\text{box}) = \text{entry in that box of } T,$$

and the same argument shows that ϕ is a linear extension.

• *From a linear extension to a standard Young tableau.* Conversely, let

$$b_1 <_L b_2 <_L \cdots <_L b_n$$

be a linear extension of P_λ, i.e. a total order on the boxes of λ compatible with the partial order. Fill the Young diagram of λ by writing 1 in b_1, 2 in b_2, and so on until we write n in b_n.

We claim that the resulting filling is a standard Young tableau. Indeed, consider two boxes in the same row, say (i, j) and (i, j') with $j < j'$. Then $(i, j) \leq (i, j')$ in P_λ, so in any linear extension we must have (i, j) appear before (i, j'), hence the number in (i, j) is smaller. Similarly, along a column, if (i, j) is above (i', j), then $(i, j) \leq (i', j)$ and the label in (i, j) is smaller. Thus rows and columns are strictly increasing, and we get a standard Young tableau.

The two constructions are clearly inverse to one another: starting from T, we recover the same total order on boxes, and starting from a linear extension, we recover the same labeling of boxes. This proves the bijection, as desired. □

Example 6.7. *For $\lambda = (2, 1)$, the Ferrers poset P_λ is the diamond poset. Proposition 6.6 tells us $e(P_\lambda)$ is the number of standard Young tableaux of shape $(2, 1)$.*

There are exactly two such tableaux:

<table><tr><td>1</td><td>2</td></tr><tr><td>3</td><td></td></tr></table>

and

<table><tr><td>1</td><td>3</td></tr><tr><td>2</td><td></td></tr></table>

(where we have merely indicated the fillings schematically). Consequently, $e(P_{(2,1)}) = 2$, matching the direct computation from the earlier example. ◦

This simple observation explains why Young diagrams show up in the study of linear extensions: each shape λ determines a poset whose linear extensions are encoded by standard Young tableaux of that shape.

Previously, we already established the hook-length formula for standard Young tableaux. Together with Proposition 6.6, this immediately answers the enumeration problem for linear extensions of Ferrers posets:

Corollary 6.8. *Let $\lambda \vdash n$ and let P_λ be the Ferrers poset associated to λ. Then:*

$$e(P_\lambda) = f^\lambda = \frac{n!}{\prod_{c \in \lambda} h(c)},$$

where $h(c)$ is the hook length of the cell c in the Young diagram of λ. †

Even in simple shapes, this result gives nontrivial closed forms.

Example 6.9 (Rectangular Ferrers posets)**.** *Let $\lambda = (k^r)$ be a rectangular partition (an $r \times k$ rectangle), so the Ferrers poset P_λ has kr elements. For the cell in row i and column j (with $1 \leq i \leq r$ and $1 \leq j \leq k$), its hook length is*

$$h(i, j) = (k - j) + (r - i) + 1 = k - j + r - i + 1.$$

Hence, the corollary gives

$$e(P_{(k^r)}) = \frac{(kr)!}{\displaystyle\prod_{i=1}^{r} \prod_{j=1}^{k} (k - j + r - i + 1)}.$$

Thus, the number of linear extensions of a rectangular Ferrers poset is encoded directly by the hook lengths of the corresponding Young diagram. ◦

So for Ferrers posets P_λ we have a remarkably clean answer to the enumeration problem for linear extensions, and the answer is – not so surprisingly – encoded by the geometry of the Young diagram λ.

6.3 P-partitions and Generating Functions (a very brief glimpse)

We finish the paper by introducing some connections and motivations of tableaux applications in Generating functions and *P-partitions*. Interested readers may check out [1] for a deeper exploration of the topic.
We start by refining the bijection between SYT's and linear extensions: Instead of labeling the boxes of P_λ by a *single* increasing sequence $1, 2, \ldots, n$, we can allow more general "weights" while still respecting the order. This leads to the theory of *P-partitions*.

We only sketch the basic idea and one key connection to Young tableaux.

Definition 6.10 (Informal)**.** Let $(P, \leq)$ be a finite poset together with a *labeling* $\omega : P \to \mathbb{Z}^+$ that assigns a positive integer to each element of P. A *(P, ω)-partition* is a map $f : P \to \mathbb{Z}^+$ such that

- if $x < y$ in P and $\omega(x) < \omega(y)$, then $f(x) \geq f(y)$,
- if $x < y$ in P and $\omega(x) > \omega(y)$, then $f(x) > f(y)$.

In words: f is order-reversing, with a weak/strict distinction depending on the labels. △

For each (P, ω)-partition f we can form a monomial

$$x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots ,$$

where m_i counts how many elements of P receive the value i under f. Summing over all (P, ω)-partitions gives a formal power series

$$K_{P,\omega}(x_1, x_2, \ldots) = \sum_f x_1^{m_1(f)} x_2^{m_2(f)} \cdots ,$$

called the *(P, ω)-partition generating function.* These generating functions turn out to be surprisingly well-structured.

Example 6.11 (Ferrers posets and semistandard Young tableaux)**.** *Let λ be a partition and P_λ its Ferrers poset. Choose a labeling ω that is compatible with the reading order of the diagram (e.g. from right to left in each row, from top row to bottom row). Then (P_λ, ω)-partitions are in natural bijection with* semistandard Young tableaux *of shape λ: fill the boxes of λ with nonnegative integers, weakly increasing along rows and strictly increasing along columns, and read those entries as the values of f. Under this bijection, the generating function $K_{P_\lambda,\omega}$ becomes*

$$K_{P_\lambda,\omega}(x_1, x_2, \ldots) = \sum_T x_1^{m_1(T)} x_2^{m_2(T)} \cdots ,$$

where the sum runs over all semistandard Young tableaux T of shape λ and $m_i(T)$ counts the number of i's in T. ∘

The right-hand side of the expression above is precisely one of the standard combinatorial definitions of the *Schur function* s_λ. Thus, for a suitable labeling,

$$K_{P_\lambda,\omega}(x_1, x_2, \ldots) = s_\lambda(x_1, x_2, \ldots).$$

We will not pursue this further here, but the idea is that:

- Young diagrams and tableaux encode linear extensions of Ferrers posets through Proposition 6.6;
- the same shapes encode richer labelings of these posets (P-partitions), whose generating functions are Schur functions;
- Schur functions, in turn, play a central role in the representation theory of both S_n and GL_n.

In this way, the modest act of counting linear extensions of a poset of boxes is already a doorway into symmetric functions and deeper representation theory.

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