Q-ANALOGUES

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1. Introduction

A q-analogue of a classical formula is a new expression depending on a parameter q that reduces to the original formula when $q \to 1$. The idea of introducing q into sums and products first appeared in the work of Euler. In his study of partitions, Euler examined infinite products such as $\prod_{n \geq 1}(1-q^n)$ and related them to generating functions. These products turned out to have many interesting properties, and many later developments in the subject can be traced back to Euler's calculations.

In the early 1800s, Gauss introduced what is now called the q-binomial theorem while studying special cases of hypergeometric series. This identity showed that many familiar binomial formulas continue to hold when ordinary integers are replaced by certain rational functions in q. Gauss's work marked one of the first times that a systematic q-analogue of a classical identity was written down.

At the beginning of the twentieth century, Ramanujan discovered a large number of new identities involving q-series. Many of these involved infinite products, continued fractions, and generating functions that could be expressed in compact q-product form. Ramanujan's work greatly expanded the theory and made it clear that q-series were not just isolated examples, but part of a much larger framework with connections to partitions and number theory.

Later developments in the mid-twentieth century showed that many basic ideas involving q-series can be organized using a small number of central objects, such as the q-integer, the q-factorial, the q-binomial coefficient, and the q-Pochhammer symbol. These objects appear in many q-identities and make it possible to write compact product formulas that generalize familiar algebraic expressions.

2. Preliminaries

We begin by defining the q-analog for n. Notice that

$$1 + q + q^2 \cdots q^n = \frac{1-q^n}{1-q}$$

Then, as q approaches 1, this expression approaches n. Thus, we can define this as the q-analog for n.

Definition 2.1. The q-analog of n is defined as $[n]_q = \frac{1-q^n}{1-q}$

We can also define q-analogs for many different combinatorial objects. For instance, we can define the q factorial as

$$[n!]_q = [n]_q[n-1]_q[n-2]_q \cdots [1]_q = \frac{1-q^n}{1-q} \cdot \frac{1-q^{n-1}}{1-q} \cdots \frac{1-q}{1-q}$$

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One question which may arise is why q should appear in combinatorial identities. Many formulas in combinatorics count objects without distinguishing them. Using q it is possible to refine these formulas by keeping track of some extra information. Here, each object will now contribute a power of q instead of just 1. This power of q often encodes something meaningful about the object. When q approaches 1, we this extra information collapses down, and we regain the original identity. For example, as we will demonstrate below, the q-factorial counts all permutations of of $[n]$, but weighted by inversions. The ordinary factorial only counts permutations. Thus, the q-factorial has extra information contained within it. It is not a random construction.

Definition 2.2. Let $\pi = \{\pi_1, \pi_2 \ldots, \pi_n\}$ be a permutation of $\{1, 2 \ldots, n\}$. Then an inversion in π is a pair (i, j) where $i < j$ and $\pi_i > \pi_j$. The number of inversions in π is denoted $\text{inv}(\pi)$.

It turns out that a good reason for using the q factorial is the combinatorial relation between it and inversions.

Theorem 2.3.

$$\sum_{\pi \in S_n} q^{inv(\pi)} = [n!]_q$$

Proof. Let

$$F_n(q) = \sum_{\pi \in S_n} q^{\text{inv}(\pi)}.$$

We prove by induction on n that $F_n(q) = [n!]_q$.

For $n = 1$, the only permutation is (1), which has 0 inversions, so

$$F_1(q) = 1 = [1]_q.$$

Now assume that $F_{n-1}(q) = [n-1]!_q$ for some $n \geq 2$. Take a permutation $\sigma \in S_{n-1}$ and form a permutation $\pi \in S_n$ by inserting n into one of the n possible positions from left to right.

If we insert n in position i (where $1 \leq i \leq n$), then there are exactly $n-i$ entries to the right of n. All of them are smaller than n, so the insertion creates exactly $n-i$ new inversions. Thus

$$\text{inv}(\pi) = \text{inv}(\sigma) + (n-i).$$

The contribution of all permutations obtained from a fixed σ is

$$q^{\text{inv}(\sigma)}(1 + q + q^2 + \cdots + q^{n-1}) = q^{\text{inv}(\sigma)}[n]_q.$$

Summing over all $\sigma \in S_{n-1}$, we get

$$F_n(q) = [n]_q \sum_{\sigma \in S_{n-1}} q^{\text{inv}(\sigma)} = [n]_q F_{n-1}(q).$$

By the induction hypothesis, $F_{n-1}(q) = [n-1]!_q$, so

$$F_n(q) = [n]_q [n-1]!_q = [n!]_q.$$

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3. q-Binomial Theorem

Continuing with this line of thought, we can also define the q-binomial coefficients.

Definition 3.1. The q binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n!]_q}{[k!]_q \cdot [(n-k)!]_q} = \frac{(1-q^n)\cdot(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$

There are multiple q-analogs for Pascal's formula.

Lemma 3.2.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

Proof. These can be proven easily. We only prove the first identity.

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

$$= \frac{(1-q^{n-1})(1-q^{n-2})\cdots(1-q^{n-k+1})}{(1-q^{k-1})(1-q^{k-2})\cdots(1-q)} + \frac{q^k(1-q^{n-1})(1-q^{n-2})\cdots(1-q^{n-k})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$

$$= \frac{(1-q^{n-1})(1-q^{n-2})\cdots(1-q^{n-k+1})[(1-q^k)+q^k(1-q^{n-k})]}{(1-q^k)(1-q^{k-1})\cdots(1-q)} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

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Similarly to the q factorial, there is a nice combinatorial identity which arises out of the q binomial coefficient.

Definition 3.3. The area of a lattice path is the number of one by one squares underneath the path.

Example.

Theorem 3.4.

$$\sum_{P \in L(a,b)} q^{area(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

Proof. Let $L(a,b)$ be the set of lattice paths from $(0,0)$ to (a,b) using steps $E = (1,0)$ and $N = (0,1)$. Define

$$A_{a,b}(q) = \sum_{P \in L(a,b)} q^{\mathrm{area}(P)}$$

We derive a recursion for $A_{a,b}(q)$ by looking at the last step of the path. Every path in $L(a,b)$ ends either with an E step or an N step.

First suppose the last step is E. Then the path can be written as $P = P'E$ where $P' \in L(a-1,b)$. The last horizontal step runs along the top of the box and does not create any new unit squares under the path, so

$$\mathrm{area}(P) = \mathrm{area}(P').$$

The contribution of all such paths is $A_{a-1,b}(q)$.

Next suppose the last step is N. Then the path can be written as $P = P'N$ where $P' \in L(a, b-1)$. The last vertical step lies in the rightmost column of the $a \times b$ rectangle. It passes above exactly a unit squares, so

$$\text{area}(P) = \text{area}(P') + a$$

The contribution of all such paths is $q^a A_{a,b-1}(q)$.

Combining these two cases, we obtain the recursion

$$A_{a,b}(q) = A_{a-1,b}(q) + q^a A_{a,b-1}(q)$$

Now define

$$B_{a,b}(q) = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

By the first q-Pascal identity,

$$\begin{bmatrix} a+b \\ a \end{bmatrix}_q = \begin{bmatrix} a+b-1 \\ a-1 \end{bmatrix}_q + q^a \begin{bmatrix} a+b-1 \\ a \end{bmatrix}_q$$

so $B_{a,b}(q)$ satisfies the same recursion

$$B_{a,b}(q) = B_{a-1,b}(q) + q^a B_{a,b-1}(q).$$

For the boundary conditions, if $b = 0$ or $a = 0$ there is only one path (all E steps or all N steps), and its area is 0, so

$$A_{a,0}(q) = A_{0,b}(q) = 1.$$

On the other hand,

$$B_{a,0}(q) = \begin{bmatrix} a \\ a \end{bmatrix}_q = 1, \quad B_{0,b}(q) = \begin{bmatrix} b \\ 0 \end{bmatrix}_q = 1$$

Thus $A_{a,b}(q)$ and $B_{a,b}(q)$ satisfy the same recursion with the same initial conditions. By induction on $a + b$, they are equal, so

$$\sum_{P \in L(a,b)} q^{\text{area}(P)} = A_{a,b}(q) = B_{a,b}(q) = \begin{bmatrix} a+b \\ a \end{bmatrix}_q.$$

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Theorem 3.5 (q-binomial Theorem)**.**

$$\prod_{k=0}^{n-1} (1 + xq^k) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^k$$

The q-binomial theorem can be viewed as a natural refinement of the classical binomial theorem. In the product

$$(1 + x)(1 + xq)(1 + xq^2) \cdots (1 + xq^{n-1}),$$

choosing the xq^k term from exactly k of the factors produces a term of the form $x^k q^{i_1+i_2+\cdots+i_k}$, where $0 \le i_1 < \cdots < i_k \le n-1$. Thus the coefficient of x^k is the generating function over all k-element subsets of $\{0, 1, \ldots, n-1\}$, weighted by the sum of the chosen indices. Shifting the indices gives a partition that fits inside a $k \times (n-k)$ rectangle, and the weight becomes $|\lambda| + k(k-1)/2$. Since the generating function for all such partitions is $\begin{bmatrix} n \\ k \end{bmatrix}$, this explains the appearance of both the Gaussian binomial coefficient and the factor $q^{k(k-1)/2}$.

Proof. Suppose $f(x) = \prod_{k=0}^{n-1}(1 + xq^k)$. Also suppose that

$$f(x) = (1 + x)(1 + xq) \cdots (1 + xq^{n-1}) = \sum_{k=0}^{n} P_k x^k$$

This means that

$$(1 + qx)f(qx) = (1 + q^n x)f(x)$$

Substituting in the sum for $f(x)$,

$$(1 + x)\sum_{k=0}^{n} P_k q^k x^k = (1 + q^n x)\sum_{k=0}^{n} P_k x^k$$

Examining the coefficients tells us that

$$P_{k-1}q^k + P_{k-1}q^{k-1} = P_k + q^n P_{k-1}$$

Then,

$$P_k = P_{k-1}\frac{q^{n-k+1} - 1}{q^k - 1}q^{k-1}$$

If we apply this repeatedly, we get that

$$P_k = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2}$$

This is proves the q-binomial theorem. ∎

4. Products and Partitions

A partition of an integer n is a decomposition of n into a sum of integers where order does not matter. For example, if we consider the number 6, then some examples of partitions are

$$1 + 1 + 1 + 3$$
$$2 + 2 + 2$$
$$1 + 1 + 4$$

Note that $1 + 3 + 1 + 1$ is not considered a distinct partition and is the same as $1 + 1 + 1 + 3$. Therefore, we write a partition with the parts in ascending order. We define the function $p(n)$ to be the number of partitions of n.

Definition 4.1. The q-Pochhammer symbol can be defined as

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$$

The q-Pochhammer symbol can be extended to an infinite product as such:

$$(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$$

The q-Pochhammer symbol is very important in partition theory, as the term $(1 - q^n)^{-1}$ encodes that n may appear a nonnegative number times. This can be seen with the geometric series expansion

$$(1 - q^n)^{-1} = 1 + q^n + q^{2n} \cdots$$

Choosing q^{kn} means that n is included exactly k times in a partition. This gives an explanation as to why the generating function for the partitions is $(q; q)_\infty^{-1}$. We also have the following

Theorem 4.2. *The number of partitions of m into exactly n parts is equal to the coefficient of $q^m a^n$ in*

$$(a;q)_\infty^{-1} = \prod_{k=0}^{\infty}(1-aq^k)^{-1}$$

Proof. To understand the product, we look at one factor at a time. For a fixed positive integer k, the expansion

$$(1-aq^k)^{-1} = 1 + aq^k + a^2q^{2k} + a^3q^{3k} + \cdots$$

lists all possible ways the part k can appear in a partition. Choosing the first term means that the part k is not used. Choosing the second term means that the part k is used once. Choosing the third term means that it is used twice, and so on. Each choice contributes a power of a equal to the number of times k is used, and a power of q equal to the total contribution of those copies to the sum of the parts.

When we take the full product over all $k \geq 1$, we are making one such choice for every possible part size. Since a partition only uses finitely many parts, almost all of these choices will be the first term, meaning the corresponding part size does not appear. After all choices are made, multiplying them together produces a monomial whose power of a is the total number of parts chosen, and whose power of q is the total sum of all parts.

A term a^nq^m therefore appears exactly when the chosen parts add up to m and there are exactly n of them. This is precisely a partition of m into n parts. Every such partition determines exactly one selection of terms in the product, and every valid selection determines exactly one partition. Thus the coefficient of a^nq^m in $(a;q)_\infty^{-1}$ counts the number of partitions of m into exactly n parts. ∎

We can express other q-analogs in terms of the q-Pochhammer symbol. For example,

$$[n!]_q = \frac{(q;q)_n}{(1-q)^n}$$

Extending this, we have the q-gamma function

$$\Gamma_q(x) = \frac{(1-q)^{1-x}(q;q)_\infty}{(q^x;q)_\infty}$$

The q-Pochhammer symbol also appears very naturally in the generating function for partitions.

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}$$

We also have that

$$\sum_{n=0}^{\infty} p_d(n)q^n = (-q;q)_\infty$$

where $p_d(n)$ is the number of partitions of n into distinct parts. These results can be used to show a famous theorem of Euler.

Theorem 4.3 (Euler)**.** *The number of partitions of a positive integer* n *into distinct parts is equal to to the number of partitions of* n *into odd parts.*

Proof.

$$\sum_{n=0}^{\infty} p_d(n)q^n = (-q;q)_\infty = \prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{1+q^{2n}}{1-q^n} = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}}$$
$$= \frac{1}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} p_o(n)q^n$$

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