DIFFERENTIAL POSETS

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Abstract. This is an expository paper on the study of differential posets, a family of posets introduced by Richard Stanley in a 1988 paper. A majority of this paper is based on Stanley's original paper, but some parts of it come from others' work on the subject.

1. Introduction and Examples

In his 1988 paper in the Journal of the American Mathematical Society, Richard Stanley introduced a kind of poset called a differential Poset.

Definition 1.1. A poset P is r**-differential** for some positive intger r if

(1) P is **locally finite**, or any interval has a finite number of elements.

(2) P is **graded** and has a **unique** minimal element O.

(3) Any two distinct $x, y \in P$ covering k elements are covered by k elements.

(4) The number of elements covering any $x \in P$ is r more than the number of elements covered by x. Equivalently, we let $C^+(x)$ be the set of elements covering x, and $C^-(x)$ be the set of elements covered by x, then $|C^+(x)| = r + |C^-(x)|$.

One example of a differential poset is Young's Lattice, with the following Hasse diagram.

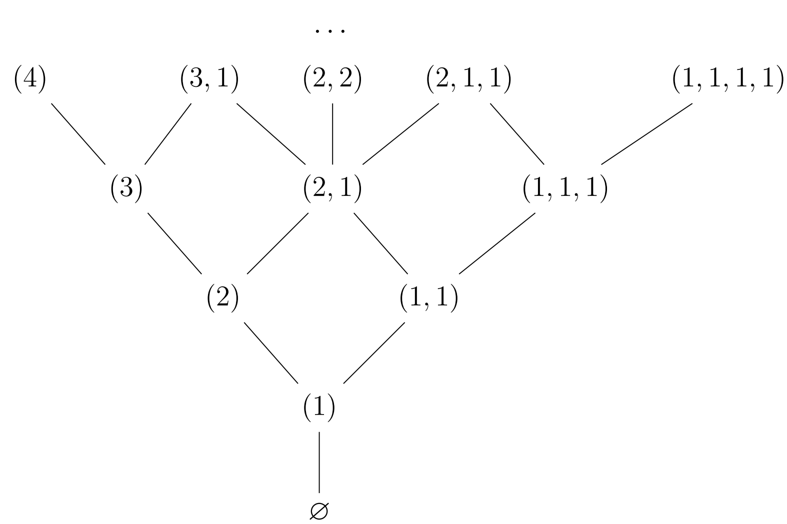

Figure 1. Young's Lattice Y of integer partitions

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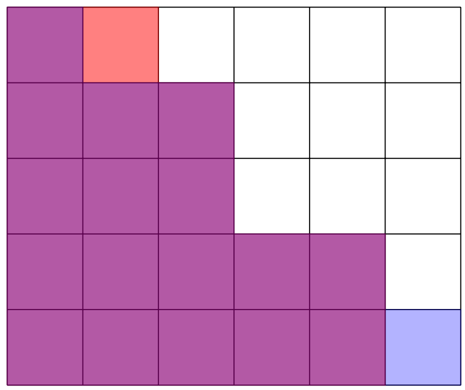
This is the lattice of integer partitions, ordered by inclusion as Young Diagrams.

Claim 1.2. *Young's Lattice is* 1*-differential.*

Proof. To show that the Young's Lattice Y is differential, we note that Y is graded where the rank function is the sum of the elements and locally finite since the partition function is finite. The minimal element is $\varnothing$, or the empty partition. The number of elements covered by a partition x with k distinct elements is k, and the number that cover it is $k+r=k+1$. The number of partitions covered by x and y is the number of ways to remove one box from x as a Young Diagram, then add one to make y. Then, if you consider the intersection of x and y, it must remain unchanged since x and y are distinct. This means that there is at most 1 box in x that is not the intersection of x and y, and similarly for y.



Here there is one tableaux that covers both the blue and red, the union of them. There is also one that is covered by both, the intersection. Whenever there is at least one of either, there is exactly one of both. This means that Y is a 1-differential poset. ∎

In fact, Young's Lattice is indeed a lattice.

Definition 1.3. A **lattice** is a poset where for any distinct elements x, y we can uniquely define

(1) $x \vee y$, or the smallest upper bound of x and y and
(2) $x \wedge y$ or the largest lower bound of x and y.

These operations are called the **join** and **meet** of x and y, respectively.

Proposition 1.4. *Young's Tableaux is a* 1*-differential lattice.*

Proof. Note that $x \vee y = x \cup y$ and $x \wedge y = x \cap y$ for all $x, y \in Y$. ∎

Another simple differential lattice is $Z(r)$, or the Fibonacci r-differential poset.

Definition 1.5. The Fibonacci r-differential poset or lattice $Z(r)$ is the set of sequences of the characters $(1_1, 1_2, \ldots, 1_r, 2)$. We refer to any of the 1_i for some i as a 1. The element x covers y in $Z(r)$ if one of

(1) The element x can be recovered from y by placing a 1 before any other 1s or
(2) The element x can be recovered from y by replacing the first 1 with a 2.

We will use the notation a^k where a is one of the letters $(1_1, 1_2, \ldots, 1_r, 2)$ and k is a nonnegative integer to represent the sequence

$$\underbrace{a, a, \ldots, a}_{k \text{ copies}}.$$

As an example, the Hasse diagram for $Z(2)$ is

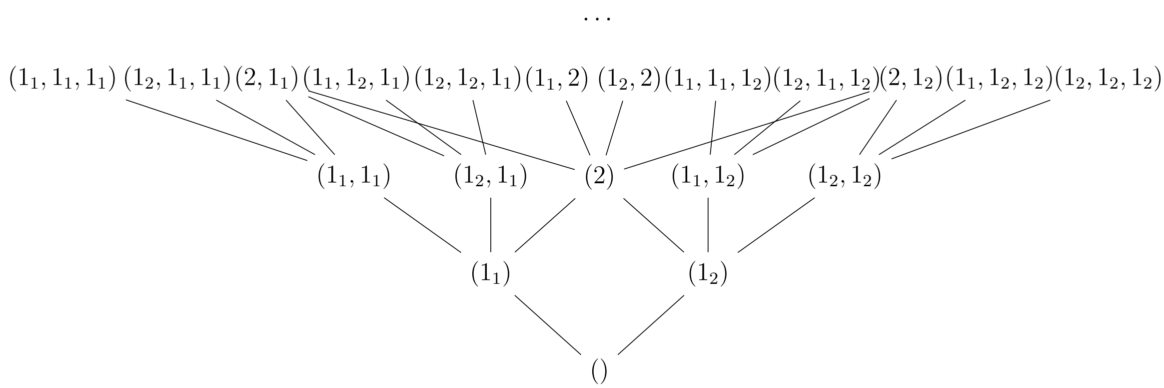


Figure 2. Fibonacci 2-differential lattice

Proposition 1.6. *The poset $Z(r)$ is r-differential. Moreover, it is a lattice and therefore modular.*

Proof. Note that 1.1 (1) and 1.1 (2) hold for $Z(r)$ with the rank function being the sum of the elements and the minimal element being the empty list. Now for 1.1 (4), consider an element $x \in Z(r)$. If $x = 2^k$ for some k, then

$$C^+(x) = \{2^m 1_n 2^{k-m} \mid 0 \leq m \leq k, 1 \leq n \leq r\}$$

and

$$C^-(x) = \{2^m 1_n 2^{k-1-m} \mid 0 \leq m \leq k-1, 1 \leq n \leq k\}.$$

Then $|C^+(x)| = kr + r$ and $|C^-(x)| = kr$, so 1.1 (4) holds for this x. Otherwise, $x = 2^k 1_n y$ for some $y \in Z(r)$. In that case,

$$C^+(x) = \{2^{k+1}y\} \cup \{2^m 1_l 2^{k-m} 1_n y \mid 0 \leq m \leq k, 1 \leq l \leq r\}$$

and

$$C^-(x) = \{2^k y\} \cup \{2^m 1_l 2^{k-1-m} y \mid 0 \leq m \leq k-1, 1 \leq l \leq r\}.$$

Then $|C^+(x)| = kr + r + 1$ and $|C^-(x)| = kr + 1$. Thus 1.1 (4) holds in $Z(r)$.

It remains to show 1.1 (3). Take two x, y that cover a common element z. Then, it must be true that ■

2. Constructing Differential Posets

Two seemingly useful questions to ask are: "Are there r-differential posets for all r?" and "Are there infinitely many?" We know the answer to the first is yes because of the powers of the Young Lattice and the Fibonacci poset. For the second, it seems useful to somehow construct r-differential posets. If we start with a finite poset that mostly satisfies the properties of a differential poset, we may want to extend it into a differential poset. To formalize this notion, we define an r-differential poset up to rank n.

Definition 2.1. A finite graded poset with a unique minimal element is **r-differential up to rank** n if it has ranks up to n and satisfies 1.1 (3) and 1.1 (4) for any elements with rank up to $n - 1$.

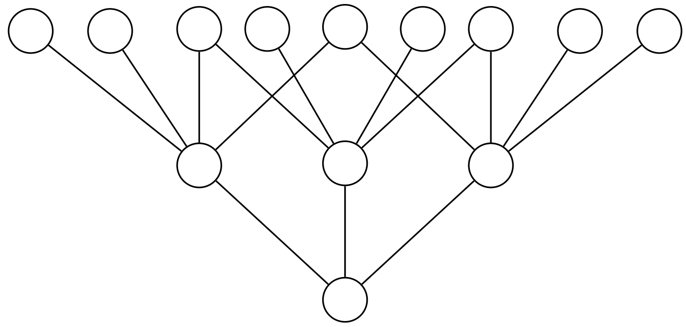


Figure 3. A 3-differential poset up to rank 2

For example, the following is a 3-differential poset up to rank 2:

We can extend any r-differential poset up to rank n into one up to rank $n+1$ by first constructing an element x' for each x of rank $n-1$ in P. Then x' covers y of rank n if and only if y covers x. Then, we can take r extra elements for each y in rank n that only cover y. This will give us an r-differential poset up to rank $n+1$, as desired. Taking the limit of this process will give us a differential poset P_∞. This process gives us two important facts.

Proposition 2.2. *From this algorithm we can find that*

(1) If P is the one-element r-differential poset O, then $P_\infty = Z(r)$ and
(2) There are infinitely many r-differential posets for any r.

Proof. For 2.2 (1), see [3], page 5. For 2.2 (2), the idea is that Young's lattice is not P_∞ for any P, so the completion of Y up to rank n will give an infinite number of distinct 1-differential posets. Then we can use 3.4 to find infinitely many r-differential posets, up to isomorphism. ■

Another useful way to think about differential posets is in terms of hypergraphs.

Definition 2.3. A **hypergraph** is a graph with the exception that an edge can connect any positive number of vertices.

Definition 2.4. The **skeleton** of a hypergraph H is a graph on the same vertex set with an edge between any two vertices that are adjacent in H.

Definition 2.5. The **dual** G^* of a hypergraph G is another hypergraph formed by swapping the vertices and edges of G.

Proposition 2.6. *An r-differential poset can be seen as a pair of sequences of hypergraphs $G_0 = (V_0, E_0), \ldots$ and $G'_0 = (V'_0, E'_0)$ such that*

- $G_n^* \cong G'_{n+1}$
- $V_n = V'_n$
- *G_n and G_{n+1} share the same skeleton*
- *the degree of v in G_n is r more than the degree of v in G'_n.*
- $|V_0| = 1$
- $E'_0 = \varnothing$.

Proof. See [1], Proposition 3.13. ■

This another definition of differential posets, and is useful for the purpose of building differential posets with hypergraphs. This can be used to find an algorithm that computes all 1-differential posets up to rank 10, which is discussed in [1] in more detail. One result of this algorithm is the number of non-isomorphic 1-differential posets up to rank n for $1 \leq n \leq 10$, which is given in the following table:

n	# of posets
1	1
2	1
3	1
4	1
5	2
6	5
7	35
8	643
9	44606
10	29199636

3. Facts about Differential Posets

Claim 3.1. *Let P be a differential poset. For any two elements $x, y \in P$, the value of k in 1.1 (3) is 0 or 1.*

Proof. Assume otherwise, or that there are two elements with $k > 1$ common covered elements. Then, take the x, y with the smallest rank with that property. Then they cover two different elements x', y'. Now x' and y' are covered by x and y, and therefore cover more than one common element. Since x' and y' have smaller rank than x, y. Thus k is always 0 or 1. ■

Definition 3.2. A **modular** lattice is one where for any three elements a, x, b with $a \leq b$, $a \vee (x \wedge b) = (a \vee x) \wedge b$.

Proposition 3.3. *Let L be a lattice satisfying all of the properties of a distributive poset except for 1.1 (3). Then it is differential if and only if it is modular.*

Proof. Given in [4] page 921. ■

Claim 3.4. *If P is an $r-$differential poset, then P^k is kr-differential for $k \geq 2$. Recall that P^k is the poset of ordered tuples $(x_1, x_2, \ldots, x_k)$ with $(x_1, x_2, \ldots, x_k)$ covering $(y_1, y_2, \ldots, y_k)$ if they differ in one component, and that component x_i covers y_i in P.*

Proof. Obviously P^k is locally finite, and its rank of $(x_1, x_2, \ldots x_k)$ is the sum of the ranks of the x_i. The unique minimal element is $(O, O, \ldots, O)$. Then, to show 1.1 (3) holds, take two distinct elements $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ in P^k. The number of elements covered by or covering both x and y is 0 unless at most 2 of the x_is are not equal to the y_is.

If exactly two of them are different, assume without loss of generality $x_1 \neq y_1$ and $y_1 \neq y_2$. Then if they both cover $z = (z_1, z_2, x_3, x_4, \ldots, x_k)$ then x_1 covers $z_1 = y_1$ and y_2 covers $z_2 = x_2$ or y_1 covers $z_1 = x_1$ and x_2 covers $y_2 = z_2$. Either way, x and y cover exactly one element and are covered by exactly one, and the two are $(x_1, y_2, x_3, \ldots, x_k)$ and $(y_1, x_2, x_3, \ldots, x_k)$ in some order. If exactly one of the components are different, then 1.1 (3) holds because it holds in P.

For 1.1 (4), note that there are r more elements covering x than there are covered by x for each component of x, so P^k is rk-differential. ■

Corollary 3.5. *If Y is Young's Lattice, Y^k is k-differential for all $k \geq 1$.*

Proof. True by 1.2 and 3.4. ■

In fact, we can generalize 3.4 as follows:

Claim 3.6. *Let P and Q be r and s-differential posets, respectively. Then $P \times Q$ is an $(r+s)$-differential poset.*

Proof. The idea is similar to that of 3.4. Obviously 1.1 (1) and 1.1 (2) is satisfied in $P \times Q$, with the minimal element being (O_P, O_Q) and the rank being the sum of the ranks in P and in Q. For an element $x = (p, q) \in P \times Q$, if p covers k elements in P and q covers m, then x covers $m + k$ elements in $P \times Q$. Since p is covered by $r + k$ elements and q is covered by $s + m$, 1.1 (4) holds. Finally, 1.1 (3) is proven in a way equivalent to in 3.4. ■

This leads to the notion of irreducible and reducible differential posets.

Definition 3.7. A differential poset P is **irreducible** if it cannot be written as the product of two differential posets, and **reducible** otherwise.

A surprising result is that the Fibonacci poset $Z(r)$ is irreducible for any r. For more information, see [4] pages $952 - 953$.

4. The D and U operators

Let KP be the vector space of linear combinations of differential poset P over a suitable field K, and let $\hat{K}P$ denote the vector space of infinite linear combinations of P over K.

We can define two transformations on $\hat{K}P$, D and U.

Definition 4.1. Let P be a locally finite poset such that for all $x \in P$, $C^+(x)$ and $C^-(x)$ are finite. For $x \in P$, Dx is the sum of the elements covered by x and Ux is the sum of the elements covering x. These can be extended to linear transformations on sums of elements of P.

Now a fundamental property of differential posets is the following:

Proposition 4.2. *Let P be a poset satisfying 1.1 (1) and 1.1 (2). Furthermore, assume that there are finitely elements of each rank. Then P is r-differential if and only if*

$$DU - UD = rI$$

where I is the identity transform.

Proof. Note that $DUx = \sum_y |C^+(x) \cap C^+(y)| y$ and $UDx = \sum_y |C^-(x) \cap C^-(y)| y$. Thus, $DU - UD = rI$ if and only if $|C^+(x) \cap C^+(y)| = |C^-(x) \cap C^-(y)|$ and $|C^+(x)| = |C^-(x)| + r$. But the first is equivalent to 1.1 (3) and the second is equivalent to 1.1 (4), and the result is proven. ■

Note that if $\frac{d}{dx}$ is the differential operator on any differentiable function $f(x)$, then $\left(\frac{d}{dx} x\right) f(x) = x \frac{d}{dx} f(x) + f(x)$, which means that

$$\frac{d}{dx} x - x \frac{d}{dx} = I,$$

akin to a 1-differential poset. If we replace $\frac{d}{dx}$ with D and x with U/r, we get the same identity as for an r-differential poset. This is the origin of the name.

Proposition 4.3. *Additionally,*

$$D\mathbf{P} = (U + rI)\mathbf{P}$$

where $\mathbf{S}$ *is the sum of the elements of* S *when* $S \in P$.

Proof. Note that the left hand side is $\sum_x C^+(x)x$ and the right hand side is $\sum_x (C^-(x)+r)x$, so they are the same by the definition of a differential poset. Note that this result holds in a poset P even if 1.1 (3) does not. ■

One tool used by Stanley is to look at formal (noncommutative) power series of U and D, or elements of the ring $K(U, D)/(DU - UD - r)$. These will not necessarily be linear transformations, such as

$$\sum_{n \geq 1} D^n,$$

but it will be helpful to look at linear transformations such as power series of U.

Definition 4.4. A **Hasse Walk** on a locally finite poset is a finite walk on the Hasse diagram of the poset.

Definition 4.5. On a differential poset P, we let $\alpha(a_1 \to a_2 \to \cdots \to a_k)$ be the number of monotone Hasse walks from an element of rank a_1 to one of rank a_2 until you reach rank a_k.

Proposition 4.6. *Let* P *be a differential poset and* β_n *be the number of sequences* $O = x_0, x_1, \ldots x_n$ *such that* x_i *covers* x_{i+1} *or is covered by* x_{i+1} *for all* $0 \leq i \leq n-1$. *Then,*

(1) The exponential generating function for $\alpha(0 \to n)$ *or*

$$\sum_{n \geq 0} \alpha(0 \to n) \frac{x^n}{n!} = \exp\left(rx + \frac{1}{2}rx^2\right).$$

(2) The exponential generating function for β_n *is*

$$\sum_{n \geq 0} \beta_n \frac{x^n}{n!} = \exp\left(rx + rx^2\right)$$

Note that both of these formulas are only dependent on r, not the actual poset P!

Proof. Found in [4], page 928. ■

This formula has a nice interpretation when you take P to be Young's Lattice Y. In that case, it holds that $\alpha(0 \to n)$ is the number of involutions on n elements. We can view a Hasse walk counted by $\alpha(0 \to n)$ as a Young Tableaux on n cells where each step is filling the next square. For example, the path

$$\varnothing \to (1) \to (1,1) \to (2,1) \to (3,1) \to (3,1,1) \to (3,2,1)$$

corresponds to the Young Tableaux

4		
3	6	
1	2	5

There is a correspondence called the Robinson–Schensted–Knuth Correspondence that bijects between standard Young Tableaux and involutions on n elements. This formula 4.6 then gives us the generating function for the number of Young Tableaux.

The use of the D and U operators give way to many more useful enumerative properties, which are explored in more depth in [4], sections 2 and 3.

5. More Recent Results and Open Problems

Stanley closed his paper [4] with 10 open problems in the field of differential posets; some of them have been solved but others haven't.

In [4], Stanley proved that rank size is weakly decreasing in a differential poset, but wasn't able to prove strongly increasing rank sizes. He also posed the question of bounds on rank size in an r-differential poset. He conjectured that the rank size is always bounded between Y^r and $Z(r)$. Byrne proved the upper bound in [1], but the lower bound remains unsolved.

Another result from [1] is that Young's Lattice and Fibonacci 1-differential lattice are the only 1-differential lattices.

For r-differential lattices, it is not known whether there are any not formed by products of Young's Lattice and some Fibonacci Lattices. In other words, there aren't known irreducible differential lattices besides Y and $Z(r)$.

Another problem proposed by Stanley is to classify the automorphism group $\text{Aut}(Z(r))$ of $Z(r)$. He wrote that this "should not be too difficult", and it was indeed solved by Jonathan Farley and Sungsoon Kim in [2].

The study of Differential Posets is relatively new, and contains many interesting results, and many more that will continue to be discovered with time.

6. Acknowledgements

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