Homomesy in Combinatorics Overview

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Abstract

In this paper, we explore homomesy across several combinatorial structures. We will focus on toggling actions on order ideals of posets (like rowmotion and promotion), their algebraic formulation through toggle groups, and many poset examples. This paper uses the style and content of works by Propp, Roby, Haddadan, and others heavily.

1 Notation

[1, 2, 3, 4, 5] Let S be a finite set, $\tau : S \to S$ a bijection (interpreted as an action or transformation), and $f : S \to \mathbb{R}$ a function, often called a statistic. Then (S, τ, f) is said to exhibit homomesy if the average of f over each τ-orbit is constant.

2 Definition and Elementary Examples

Definition 2.1 (Homomesy)**.** Let S be a finite set, $\tau : S \to S$ a bijection, and $f : S \to \mathbb{R}$. We say that (S, τ, f) is homomesic if there exists a constant c such that for every orbit $O \subset S$ under τ, we have:

$$\frac{1}{|O|} \sum_{x \in O} f(x) = c.$$

A common belief is that f is constant on each orbit, but this isn't true. Homomesy is much weaker since it only requires constancy of orbit averages.

2.1 Binary Strings Under Rotation

Let $S_{n,k}$ be the set of binary strings of length n with exactly k ones. Define a rotation map τ by moving the last bit to the front:

$$\tau(s_1 s_2 \dots s_n) = s_n s_1 s_2 \dots s_{n-1}.$$

Let $f : S_{n,k} \to \mathbb{Z}$ be the inversion number:

$$\text{inv}(s) = \#\{(i < j) : s_i = 1, s_j = 0\}.$$

Theorem 2.1 (Propp–Roby)**.** *$(S_{n,k}, \tau, inv)$ exhibits homomesy. That is,*

$$\frac{1}{|O|} \sum_{s \in O} inv(s) = \frac{k(n-k)}{2},$$

for all orbits O of τ.

Example 2.1. Take $n = 5$, $k = 2$, so that $|S_{5,2}| = 10$. The orbit of the string $s = 11000$ is:

$$11000, 01100, 00110, 00011, 10001.$$

The inversion numbers are $6, 4, 2, 0, 4$ respectively. Their average is $(6 + 4 + 2 + 0 + 4)/5 = 3.2 = 2(5 - 2)/2$.

This case shows that cyclic group actions can stabilize certain statistics in orbit average. This behavior generalizes far beyond binary strings.

3 Posets and Order Ideals

Let P be a finite poset. An order ideal $I \subseteq P$ is a subset such that if $x \in I$ and $y \leq x$, then $y \in I$. Let $J(P)$ be the set of all order ideals of P. The set $J(P)$ has a lot of possible operations, but the most important for this paper is rowmotion.

3.1 Rowmotion

Definition 3.1 (Rowmotion)**.** Let $I \in J(P)$. Define $\mathrm{row}(I)$ as the order ideal generated by the minimal elements of $P \setminus I$.

This means that we remove the elements of I from P, collect the minimal remaining elements, and form the ideal generated by them. This defines a bijection $\mathrm{row} : J(P) \to J(P)$.

3.2 Homomesy for Size Function

Let $f : J(P) \to \mathbb{Z}$ be defined by $f(I) = |I|$.

Theorem 3.1 (Propp–Roby)**.** *For $P = [a] \times [b]$, the function $f(I) = |I|$ is homomesic under rowmotion. That is,*

$$\frac{1}{|O|} \sum_{I \in O} |I| = \frac{ab}{2},$$

for all rowmotion orbits O.

Example 3.1. Let $P = [2] \times [2]$ with Hasse diagram forming a 2×2 grid. There are 6 order ideals. Rowmotion cycles through them in orbits of length 4 and 2. The total number of elements summed over an orbit divided by its size equals 2, matching $ab/2 = 2$.

In general, the order ideal size statistic is homomesic for many poset families under rowmotion.

4 Toggles and the Toggle Group

Definition 4.1 (Toggle)**.** For each $p \in P$, define a map $t_p : J(P) \to J(P)$ as follows:

$$t_p(I) = \begin{cases} I \cup \{p\}, & \text{if } p \notin I \text{ and } I \cup \{p\} \in J(P), \\ I \setminus \{p\}, & \text{if } p \in I \text{ and } I \setminus \{p\} \in J(P), \\ I, & \text{otherwise.} \end{cases}$$

Each toggle t_p is an involution: $t_p^2 = \text{id}$. The toggle group $\mathcal{T}(P)$ is the group generated by the toggles $\{t_p\}_{p \in P}$.

Rowmotion can be written as a product of toggles applied in a certain order, which we usually determine by a linear extension of P. That is,

$$\text{row} = t_{p_1} t_{p_2} \dots t_{p_n},$$

where $p_1 < p_2 < \cdots < p_n$ is a linear extension.

The toggle perspective allows us to do algebraic manipulations and make many generalizations

5 File Statistics and Comotion

Let $P = [a] \times [b]$, and define the k-th file to be the set

$$F_k = \{(i, j) \in P : i - j = k\}.$$

Define $f_k : J(P) \to \mathbb{Z}$ by $f_k(I) = |I \cap F_k|$. That is, f_k counts how many elements in file k lie in the order ideal I.

Theorem 5.1 (Haddadan)**.** *Each f_k is homomesic under rowmotion on $J([a] \times [b])$.*

This theorem shows that not only is the total size $|I|$ homomesic, but so is its size inside each diagonal file. This leads to a finer decomposition of homomesy.

6 Promotion and Comotion

6.1 Promotion

We define promotion as applying toggles in column order instead of in a linear extension. Rowmotion toggles each element only once in a specific order while promotion chooses a different ordering.

Definition 6.1 (Promotion on $J(P)$)**.** Let $P = [a] \times [b]$. Fix a total order on the columns and define pro as the composition of toggles t_p in that order.

Promotion, like rowmotion, defines a bijection on $J(P)$ with rich orbit structure. The orbit sizes for promotion are generally different but often coincide in size with those of rowmotion.

6.2 Comotion and Generalizations

Haddadan talks about comotion, which is a toggle-based dynamical system encompassing rowmotion, promotion, and more. The main idea we will explore is how to compose toggles corresponding to subsets of the poset.

Theorem 6.1 (Haddadan)**.** *Let P be a poset of type $Q_{a,b}, L_a$, or U_a, and f a linear combination of file statistics. Then f is homomesic under comotion.*

These overall ideas help explain why so many statistics exhibit homomesy under different toggle dynamics.

7 Conclusion

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References

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