Euler Circle: Ehrhart Polynomials

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Abstract

In 1899, Georg Pick discovered a simple formula relating the area of a 2-dimensional lattice polygon to the number of integer points on its boundary and in its interior. However, it fails to generalize to three or more dimensions. This paper explores the question: "What happens in higher dimensions?" We will demonstrate the failure of a simple generalization using the Reeve tetrahedron, and then introduce a new perspective: dilation. This leads us to the theory of Ehrhart polynomials, which counts lattice points in scaled versions of a polytope. We will show that these polynomials not only generalize the concept of volume but also, through a property called Ehrhart-Macdonald Reciprocity, contain Pick's original formula as a special case.

1 Introduction: Pick's Theorem

We begin in the 2-dimensional integer lattice, $\mathbb{Z}^2$, which is the set of all points (x, y) where x and y are integers.

A **lattice polygon** P is a simple polygon (one that does not intersect itself) in the xy-plane whose vertices are all points in the integer lattice $\mathbb{Z}^2 2$.

Given a lattice polygon P, we are interested in two numbers:

- $I(P)$: The number of lattice points strictly in the **interior** of P.
- $B(P)$: The number of lattice points on the **boundary** of P.

In 1899, Georg Pick discovered a formula that connects these two numbers to the area of the polygon, $A(P)$.

[**Pick's Theorem**] For any simple lattice polygon P, its area $A(P)$ is given by:

$$A(P) = I(P) + \frac{B(P)}{2} - 1$$

Let's test this theorem with an example. Consider the triangle T with vertices at $(0, 0)$, $(2, 0)$, and $(0, 1)$.

We can find the area using standard geometry: $A(T) = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 2 \times 1 = 1$.

Now, let's count the lattice points:

- **Interior points** $I(T)$**:** There are no lattice points strictly inside the triangle. So, $I(T) = 0$.
- **Boundary points** $B(T)$**:** The lattice points on the boundary are $(0, 0), (1, 0), (2, 0), (0, 1)$. Note that $(1, 1)$ and $(2, 1)$ are not on the boundary. So, $B(T) = 4$.

We check using Pick's theorem:

$$I(T) + \frac{B(T)}{2} - 1 = 0 + \frac{4}{2} - 1 = 2 - 1 = 1$$

The formula $A(T) = 1$ matches our geometric calculation. This formula begs a question: What happens in higher dimensions?

2 The Problem: Failure in 3D

It is natural to ask if a similar formula exists for 3D lattice polytopes. We might hope to find some simple linear relation between the Volume V, the number of interior points I, and the number of boundary points B:

$$V = c_1 I + c_2 B + c_3 \quad \text{(A hypothetical 3D Pick's formula)}$$

Unfortunately, no such simple formula exists. The classic counterexample is the Reeve tetrahedron.

For any integer $h \geq 1$, the **Reeve tetrahedron** R_h is the tetrahedron in $\mathbb{Z}^3$ with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, h)$.

Let's analyze the properties of R_h.

- **Lattice Points:** For any $h \geq 1$, the Reeve tetrahedron R_h has no interior lattice points. Furthermore, the only lattice points on its boundary are the four vertices. (Proving this is a good exercise, but we will take it as given). Thus, for any h, we have $I(R_h) = 0$ and $B(R_h) = 4$.
- **Volume:** The volume of a tetrahedron with one vertex at the origin and others at $\vec{a}$, $\vec{b}$, $\vec{c}$ is $V = \frac{1}{6}|\det(\vec{a}, \vec{b}, \vec{c})|$. For R_h, we have $\vec{a} = (1, 0, 0)$, $\vec{b} = (0, 1, 0)$, and $\vec{c} = (1, 1, h)$.

$$V(R_h) = \frac{1}{6}\left|\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & h \end{pmatrix}\right| = \frac{1}{6}|1(h - 0)| = \frac{h}{6}$$

Here is the problem: our hypothetical 3D Pick's formula would depend on I and B. But for all $h \geq 1$, I and B are constant ($I = 0, B = 4$). This would imply the volume must also be constant.

$$V(R_h) = c_1(0) + c_2(4) + c_3 = 4c_2 + c_3$$

But we just showed $V(R_h) = h/6$, which clearly depends on h! This is a contradiction. We conclude that no simple generalization of Pick's Theorem exists. To answer our question from above, we will need a more powerful tool.

3 Dilation and the Ehrhart Function

The failure of the hypothetical formula in 3D motivates a different approach. Instead of looking at a single polytope, what if we look at an entire family of them?

Let P be a d-dimensional polytope (e.g., a polygon in 2D or a polyhedron in 3D). For any positive integer t, the **t-th dilate** of P, denoted tP, is the polytope scaled by a factor of t:

$$tP = \{t \cdot \vec{x} \mid \vec{x} \in P\}$$

This leads us to a new way of counting. Instead of just counting points in P, let's count points in tP for all $t = 1, 2, 3, \ldots$

Let P be a d-dimensional lattice polytope. The **Ehrhart function** of P, denoted $L_P(t)$, is a function of t that counts the number of integer lattice points in the t-th dilate of P.

$$L_P(t) = |tP \cap \mathbb{Z}^d| \quad \text{for } t \in \mathbb{N}$$

Let's find the Ehrhart function for a simple polytope: the unit square $S = [0,1]^2$ in $\mathbb{Z}^2$. Its vertices are $(0,0), (1,0), (0,1), (1,1)$.

- For $t = 1$, $1S = [0,1]^2$. The lattice points are $(0,0), (1,0), (0,1), (1,1)$. There are $(1+1) \times (1+1) = 4$ points. So, $L_S(1) = 4$.
- For $t = 2$, $2S = [0,2]^2$. The lattice points (x, y) have $x \in \{0,1,2\}$ and $y \in \{0,1,2\}$. There are $(2+1) \times (2+1) = 9$ points. So, $L_S(2) = 9$.
- For $t = 3$, $3S = [0,3]^2$. The lattice points (x, y) have $x \in \{0,1,2,3\}$ and $y \in \{0,1,2,3\}$. There are $(3+1) \times (3+1) = 16$ points. So, $L_S(3) = 16$.

Looking at the sequence $4, 9, 16, \ldots$, we can make an educated guess to the formula:

$$L_S(t) = (t+1)^2 = t^2 + 2t + 1$$

Notice something remarkable: this function, which counts discrete points, turns out to be a simple polynomial.

4 Ehrhart's Theorem

This observation from our example is not a coincidence. It results from a powerful theorem by Eugène Ehrhart, published in 1962.

[**Ehrhart's Theorem**] Let P be a d-dimensional **convex** lattice polytope. The Ehrhart function $L_P(t)$ is a polynomial in t of degree d.

$$L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0$$

This polynomial is called the **Ehrhart polynomial** of P.

This theorem tells us that the counting of points inside scaled polytopes is governed by a clean polynomial. The answer lies in the polynomial's coefficients.

[**Properties of Ehrhart Coefficients**] Let $L_P(t) = c_d t^d + \cdots + c_0$ be the Ehrhart polynomial for a d-dimensional lattice polytope P.

- The leading coefficient, c_d, is the volume of P.
- The second coefficient, c_{d-1}, is half the "normal" surface volume (a bit complex, but for $d = 2$, it is $\frac{1}{2}B(P)$).
- The constant term, c_0, is $L_P(0) = 1$. (Note: This property holds for the **convex** polytopes to which this theorem applies).

Let's revisit our unit square example $S = [0, 1]^2$, where $d = 2$. We found $L_S(t) = t^2 + 2t + 1$.

- $c_2 = 1$. The volume (area) of the unit square is $1 \times 1 = 1$. It matches.
- $c_1 = 2$. The boundary of the square has 4 lattice points: $(0, 0), (1, 0), (0, 1), (1, 1)$. The boundary **length** is 4. The coefficient is $c_1 = \frac{1}{2} \times (\text{Boundary Length}) = \frac{1}{2} \times 4 = 2$. It matches.
- $c_0 = 1$. It matches.

So, the Ehrhart polynomial $L_P(t)$ encodes the Volume of P (as c_d) and its boundary measure (as c_{d-1}). This is the generalization we were looking for.

5 Ehrhart-Macdonald Reciprocity

So far, we've been counting **all** the lattice points, but Pick's formula cared about **interior** points too. Let's define a polynomial for them.

Let P be a lattice polytope. We define $I_P(t)$ as the number of **interior** lattice points in the t-th dilate of P.

$$I_P(t) = |\text{int}(tP) \cap \mathbb{Z}^d|$$

This function $I_P(t)$ also turns out to be a polynomial.

But how do $L_P(t)$ (all points) and $I_P(t)$ (interior points) relate?

[**Ehrhart-Macdonald Reciprocity**] Let P be a d-dimensional lattice polytope. The polynomials $L_P(t)$ and $I_P(t)$ are related by the following simple equation:

$$I_P(t) = (-1)^d L_P(-t)$$

This theorem states that if you take the polynomial that counts all lattice points and evaluate it at **negative** integers, you get back the number of **interior** points for positive integers. It's a "reciprocity" that links the inside and the outside of the polyhedron.

For our unit square S, $d = 2$ and $L_S(t) = t^2 + 2t + 1$. Reciprocity predicts:

$$I_S(t) = (-1)^2 L_S(-t) = L_S(-t) = (-t)^2 + 2(-t) + 1 = t^2 - 2t + 1 = (t - 1)^2$$

Let's check this. $I_S(t)$ counts interior points in $tS = [0, t]^2$. The interior is $(0, t) \times (0, t)$.

- $t = 1$: $I_S(1) = (1 - 1)^2 = 0$. Interior of $[0, 1]^2$ is $(0, 1)^2$. It has 0 lattice points. This is correct.

- $t = 2$: $I_S(2) = (2-1)^2 = 1$. Interior of $[0,2]^2$ is $(0,2)^2$. It has one lattice point: $(1,1)$.
- $t = 3$: $I_S(3) = (3-1)^2 = 4$. Interior of $[0,3]^2$ is $(0,3)^2$. It has four lattice points: $(1,1),(1,2),(2,1),(2,2)$.

So, the reciprocity theorem works.

6 Rederiving Pick's Formula

We now have all the tools to answer our original question. We can use Ehrhart polynomials to derive Pick's 2D formula, showing that Pick's is just a piece of a much larger picture.

Let P be any 2D lattice polygon ($d = 2$). From Ehrhart's Theorem, we know:

$$L_P(t) = c_2t^2 + c_1t + c_0$$

From the properties of the coefficients:

- $c_2 = \text{Volume}(P) = A(P)$ (the Area).
- $c_1 = \frac{1}{2}B(P)$ (for $d = 2$, this is half the number of boundary points).
- $c_0 = 1$.

Substituting these into the polynomial, we get:

$$L_P(t) = A(P)t^2 + \frac{B(P)}{2}t + 1$$

Now, let's use Ehrhart-Macdonald Reciprocity to find the interior polynomial, $I_P(t)$:

$$I_P(t) = (-1)^2L_P(-t) = L_P(-t)$$

$$I_P(t) = A(P)(-t)^2 + \frac{B(P)}{2}(-t) + 1 = A(P)t^2 - \frac{B(P)}{2}t + 1$$

We now have two equations for $A(P)$, $B(P)$, and $I(P)$. All we have to do is evaluate them at $t = 1$.

- $L_P(1)$ is, by definition, the total number of lattice points in $1P = P$. This is $I(P) + B(P)$.
- $I_P(1)$ is, by definition, the total number of interior lattice points in $1P = P$. This is $I(P)$.

Let's use the second formula, $I_P(t)$, and set $t = 1$:

$$I_P(1) = A(P)(1)^2 - \frac{B(P)}{2}(1) + 1$$

$$I(P) = A(P) - \frac{B(P)}{2} + 1$$

Now, we just rearrange the terms to solve for $A(P)$:

$$A(P) = I(P) + \frac{B(P)}{2} - 1$$

This is exactly Pick's Theorem.

7 Conclusion

We began with a simple 2D formula, Pick's Theorem, and asked what happens in higher dimensions. We quickly saw that a simple generalization fails, forcing us to adopt a new perspective. By counting lattice points in **dilations** of a polytope P, we discovered the Ehrhart polynomial $L_P(t)$.

This polynomial not only encodes the d-dimensional volume as its leading coefficient but also obeys reciprocity that relates all lattice points to just the interior ones. From this higher dimensional theory, Pick's original formula comes out as a simple $d = 2, t = 1$ case.

References

[1] M. Beck and S. Robins. *Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra.* Springer, 2007.

[2] E. Ehrhart. Sur les polyèdres rationnels et les réseaux. *J. Reine Angew. Math.*, 227:25–49, 1967.

[3] G. Pick. Geometrisches zur Zahlenlehre. *Sitzungsber. Lotos (Prag)*, 19:311–319, 1899.