PATTERN AVOIDANCE IN PERMUTATIONS AND RELATED STRUCTURES

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ABSTRACT. We examine pattern avoidance in permutations of length three. We prove that all such patterns are Wilf-equivalent, enumerated by the Catalan numbers, and establish bijections to Dyck paths and stack-sorting.

1. Introduction

The study of permutation patterns allows us to classify permutations based on their internal structure. A permutation is said to contain a pattern if it has a subsequence that appears in the same relative order as the pattern. If it does not contain the pattern, we say it *avoids* it.

While the definition is simple, the counting of pattern-avoiding permutations connects to diverse combinatorial objects such as binary trees, Dyck paths, and Young tableaux. As shown in *Combinatorics*, these connections often reveal that seemingly unrelated problems are counted by the same sequence of numbers, most notably the Catalan numbers [1].

In this paper, we focus on patterns of length three. We prove that all six patterns of length three are Wilf-equivalent. That is, for any n, the number of permutations of length n avoiding a specific pattern $\sigma \in S_3$ is the same regardless of which σ is chosen. We establish this result using symmetries and recursion. We then present the bijection between these permutations and Dyck paths, and clarify the relationship between pattern avoidance and stack-sorting. We conclude by discussing real-world applications and how these concepts extend to cyclic permutations [3] and integer compositions [2].

2. Definitions and Symmetries

2.1. Basic Definitions.

Definition 2.1. A **permutation** of length n is a bijection from $\{1, \ldots, n\}$ to itself. We typically write a permutation π in one-line notation as $\pi = \pi_1\pi_2 \cdots \pi_n$. The set of all permutations of length n is denoted $\mathcal{S}_n$.

Definition 2.2. A permutation $\pi \in \mathcal{S}_n$ **contains** a pattern $\sigma \in \mathcal{S}_k$ if there exists a set of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the subsequence $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}$ is order-isomorphic to σ. That is, for all x, y, $\pi_{i_x} < \pi_{i_y}$ if and only if $\sigma_x < \sigma_y$. If π does not contain σ, we say π **avoids** σ. The set of permutations in $\mathcal{S}_n$ avoiding σ is denoted $\mathrm{Av}_n(\sigma)$, and we let $s_n(\sigma) = |\mathrm{Av}_n(\sigma)|$.

Example 2.3. Consider $\pi = 3\,1\,4\,2$.

- Does it contain 123? We look for three entries increasing left-to-right. The only increasing subsequences are $3, 4$ and $1, 4$ and $1, 2$. None have length 3. Thus, π avoids 123.

- Does it contain 132? We look for "small, large, medium". The subsequence $1, 4, 2$ (indices 2, 3, 4) fits this pattern perfectly ($1 < 2 < 4$). Thus, π contains 132.

2.2. **Symmetries and Wilf Equivalence.** Two patterns σ and τ are called **Wilf-equivalent** if $s_n(\sigma) = s_n(\tau)$ for all $n \geq 0$. While one could compute the sequences s_n for various n to check for equivalence, a better method is to use the natural symmetries of the square on which the permutation matrices are drawn.

There are three primary operations that preserve pattern avoidance behavior [4]:

(1) **Reversal** (r): Reading the permutation backwards.

$$r(\pi) = \pi_n \pi_{n-1} \cdots \pi_1$$

(2) **Complement** (c): Replacing each value x with $n + 1 - x$.

$$c(\pi) = (n + 1 - \pi_1)(n + 1 - \pi_2) \cdots (n + 1 - \pi_n)$$

(3) **Inverse** (π^{-1}): The usual group-theoretic inverse.

Proposition 2.4. *For any pattern σ, $s_n(\sigma) = s_n(r(\sigma)) = s_n(c(\sigma)) = s_n(\sigma^{-1})$.*

Proof. These operations are bijections on $\mathcal{S}_n$. For example, π avoids σ if and only if $r(\pi)$ avoids $r(\sigma)$. Thus, the map $\pi \mapsto r(\pi)$ is a bijection from $\mathrm{Av}_n(\sigma)$ to $\mathrm{Av}_n(r(\sigma))$. The same logic applies to complement and inverse. □

Let us apply these to $\mathcal{S}_3$:

$$123, \quad 132, \quad 213, \quad 231, \quad 312, \quad 321$$

Using the complement c:

- $c(123) = 321$
- $c(132) = 312$
- $c(213) = 231$

Using the reversal r:

- $r(132) = 231$

From these basic operations, we can form equivalence classes. 132 is equivalent to 312 (via complement) and 231 (via reversal). 231 is equivalent to 213 (via inverse). Thus, the patterns $\{132, 213, 231, 312\}$ are all Wilf-equivalent. Separately, 123 is equivalent to 321 via reversal.

This partitions $\mathcal{S}_3$ into two classes: $\{123, 321\}$ and $\{132, 213, 231, 312\}$. To prove that *all* six are equivalent, we must connect these two classes. We do this by showing that $s_n(123) = s_n(132)$. This is a non-trivial result because 123 and 132 are not related by simple symmetries. Instead, their equivalence is established because both count sequences satisfy the same Catalan recurrence relation, as we prove in the next section.

3. Enumeration of 132-Avoiding Permutations

We now derive the formula for $s_n(132)$. We use a recursive approach based on the position of the largest element, n.

Theorem 3.1. *For all $n \geq 0$, $s_n(132) = C_n = \frac{1}{n+1}\binom{2n}{n}$.*

Proof. Let $\pi \in \mathrm{Av}_n(132)$. Suppose the largest element n is at index k, so $\pi_k = n$. We can write π as:

$$\pi = \alpha\, n\, \beta$$

where α is the prefix of length $k-1$ and β is the suffix of length $n-k$.

For π to avoid 132, two conditions must be met:

(1) α and β must individually avoid 132.
(2) No element in α can be smaller than an element in β.

Justification for (2): Suppose there existed $a \in \alpha$ and $b \in \beta$ such that $a < b$. Since $b \in \beta$, it appears after n. Since $a \in \alpha$, it appears before n. Also, n is the maximum, so $b < n$. Thus, the indices of a, n, b form the values $a < b < n$ in the order "$a \dots n \dots b$". This corresponds to the pattern 132 (Smallest, Largest, Middle). This contradicts the assumption that π avoids 132.

Therefore, all elements in α must be larger than all elements in β. Since the set of values in π is $\{1, \dots, n\}$ and $\pi_k = n$, the set of values in α must be $\{n-k+1, \dots, n-1\}$ and the set of values in β must be $\{1, \dots, n-k\}$.

Once the values are partitioned this way, the structure of α is order-isomorphic to a permutation in $\mathrm{Av}_{k-1}(132)$, and β is order-isomorphic to a permutation in $\mathrm{Av}_{n-k}(132)$. These choices are independent. Summing over all possible positions k of n:

$$s_n(132) = \sum_{k=1}^{n} s_{k-1}(132)\, s_{n-k}(132).$$

With the base case $s_0(132) = 1$ (the empty permutation), this is the defining recurrence relation for the Catalan numbers C_n. Since 123-avoiding permutations also satisfy this recurrence (by a different structural decomposition), we conclude $s_n(123) = s_n(132)$, completing the proof that all length-3 patterns are Wilf-equivalent. □

3.1. **Generating Functions.** Let $A(x) = \sum_{n=0}^{\infty} s_n(132)x^n$. Using the recurrence derived above:

$$A(x) = 1 + x \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} s_{k-1}s_{n-k} \right) x^{n-1} = 1 + xA(x)^2.$$

Solving for $A(x)$ gives the generating function for the Catalan numbers:

$$A(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

The coefficient of x^n in the expansion of this series is well-known to be C_n.

Since $s_n(123)$ also equals C_n, we conclude that all patterns in $\mathcal{S}_3$ are Wilf-equivalent [1].

4. Bijective Connections

One of the best aspects of pattern avoidance is the ability to map these permutations to other structures. We present two fundamental bijections.

4.1. **Dyck Paths and 132-Avoidance.** A Dyck path of semilength n is a lattice path from $(0,0)$ to $(2n,0)$ consisting of Up (U) and Down (D) steps that never goes below the x-axis.

We define a bijection $\Phi : \mathrm{Av}_n(132) \to \mathcal{D}_n$ recursively, matching the recursive structure used in Theorem 3.1. Let $\pi \in \mathrm{Av}_n(132)$.

(1) If π is empty, $\Phi(\pi)$ is the empty path.
(2) If $\pi = \alpha\, n\, \beta$, where all elements in α are greater than elements in β, then:

$$\Phi(\pi) = U\,\Phi(\alpha)\,D\,\Phi(\beta).$$

Example 4.1. Consider the permutation $\pi = 2\,1\,3$. This permutation avoids 132. To find its corresponding Dyck path, we identify the largest element $n = 3$, which is at index $k = 3$. This splits π into a prefix $\alpha = 2\,1$ and a suffix $\beta = \emptyset$. Note that $\alpha > \beta$, satisfying the condition. We compute $\Phi(213) = U\,\Phi(21)\,D\,\Phi(\emptyset)$ recursively:

- **Step 1 (Main Split):** The max element is 3. $\alpha = 21$, $\beta = \emptyset$.

$$\Phi(213) = U\,\Phi(21)\,D\,\Phi(\emptyset).$$

- **Step 2 (Recurse on $\Phi(21)$):** For the sequence $2\,1$, the max element is 2 (at index 1 relative to the sequence). This splits it into prefix $\alpha' = \emptyset$ and suffix $\beta' = 1$.

$$\Phi(21) = U\,\Phi(\emptyset)\,D\,\Phi(1).$$

- **Step 3 (Base Cases):** We know $\Phi(\emptyset)$ is the empty path ϵ. For $\Phi(1)$, the max is 1, so $\Phi(1) = U\epsilon D\epsilon = UD$.
- **Combining results:**

$$\Phi(21) = U(\epsilon)D(UD) = UDUD.$$

$$\Phi(213) = U(UDUD)D(\epsilon) = UUDUDD.$$

The result is the Dyck path $UUDUDD$, illustrated in Figure 1.

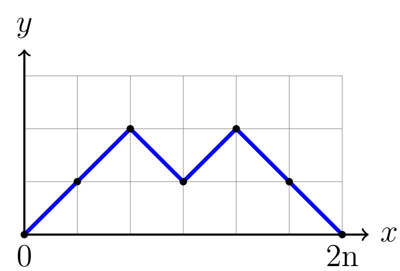


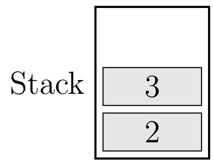
FIGURE 1. The Dyck path $UUDUDD$ corresponding to $\pi = 213$.

4.2. **Stack Sorting and 231-Avoidance.** Historically, the study of pattern avoidance began with Knuth's analysis of stack-sortable permutations. A stack is a Last-In-First-Out (LIFO) data structure. We want to sort a permutation π into the identity $123\ldots n$ using a stack.

The algorithm is:

(1) Attempt to push the next element of the input permutation onto the stack.
(2) If the top of the stack is the next number needed for the output (1, then 2, etc.), pop it.

Theorem 4.2 (Knuth)**.** *A permutation π is stack-sortable if and only if π avoids the pattern* 231*.*



Blocking Mechanism:
The value 3 is on top of 2.
To sort, we need to pop 2 before 3.
But 3 blocks 2. Thus, the pattern 231 fails to sort.

FIGURE 2. Visualizing the stack conflict caused by a 231 pattern.

Proof. ($\Rightarrow$) The condition is necessary. Suppose π contains a 231 pattern. Then there exist indices $i < j < k$ such that $\pi_k < \pi_i < \pi_j$. Let these values be $2, 3, 1$ (relative order). When the algorithm processes π, the element corresponding to 3 (π_j) enters the stack. At this point, the element corresponding to 2 (π_i) is already in the stack (since $i < j$) and has not been popped (since it must wait for 1 (π_k) to be output first, as $1 < 2$). Thus, 3 sits above 2 in the stack. However, to sort the permutation, 2 must be popped before 3. This is impossible because 3 blocks 2. Thus, π is not sortable.

($\Leftarrow$) The condition is sufficient. We prove that if π avoids 231, it is sortable. We proceed by induction on n. Let π be a 231-avoiding permutation and let n be the largest element. Since π avoids 231, we cannot have any elements to the left of n that are larger than any elements to the right of n (otherwise we form a $x \dots n \dots y$ pattern with $x > y$, which is 231). Thus, π must have the structure $\pi = L\,n\,R$, where every element in L is smaller than every element in R. The algorithm processes L first. Since elements in L are small, they are all pushed and eventually popped (sorted) before n is processed, by the inductive hypothesis. The stack is then empty. Next, n is pushed. Then R is processed. Since all elements in R are smaller than n (but larger than L), they will be processed and sorted while n sits at the bottom of the stack. Finally, n is popped. The result is the sorted sequence $L_{sorted}\,R_{sorted}\,n$, which is $1 \dots n$. ☐

5. Variations and Generalizations

5.1. **Cyclic Permutations.** Standard permutations are linear. However, one can view a permutation as an arrangement of numbers on a circle. Vella [3] and others have studied pattern avoidance in this context.

A **cyclic permutation** $[\pi]$ is the equivalence class of π under rotation. Avoidance in cyclic terms is stricter. A cyclic permutation $[\pi]$ avoids a pattern σ if *every* linear rotation of π avoids σ.

Theorem 5.1 (Vella [3])**.** *The set* $\mathrm{Av}_n^c(321)$ *of cyclic permutations avoiding* 321 *is in bijection with non-decreasing Dyck paths.*

Proof Sketch. Consider a cyclic permutation $[\pi]$ that avoids 321. We can uniquely represent $[\pi]$ by the linear permutation in its rotation class that starts with n. Let $\pi = n\sigma_1 \dots \sigma_{n-1}$. Since $[\pi]$ avoids 321, the linear permutation $\sigma_1 \dots \sigma_{n-1}$ must avoid 21. This implies that the sequence σ must be increasing, which is a very strong, seemingly trivial restriction. However, the condition is that *all* rotations must avoid 321, not just the one starting with n. Vella constructs a bijection by mapping the permutation to a Dyck path. The requirement that *every* rotation avoids 321 imposes a restriction on the "valleys" of the corresponding path. Specifically, if a rotation contained a 321, it would correspond to a valley in the Dyck path dipping lower than a previous valley in a specific way. Avoiding this across all rotations forces the heights of the valleys to be non-decreasing. This subset of Dyck paths is enumerated by a more complex sequence than the Catalan numbers. □

5.2. **Pattern Avoidance in Compositions.** Savage and Wilf extended the notion of pattern avoidance from permutations to **compositions** [2]. A composition of an integer n is a sequence of positive integers summing to n. Unlike permutations, elements can be repeated.

Theorem 5.2 (Savage-Wilf [2])**.** *The generating function $C(x)$ for the number of compositions of n avoiding the pattern 123 is:*

$$C(x) = \frac{1-x}{1-2x+x^3}.$$

Derivation. Let $C(n)$ be the number of 123-avoiding compositions of n. A composition avoids 123 if it contains no three parts c_i, c_j, c_k (indices $i < j < k$) such that $c_i \leq c_j \leq c_k$. Savage and Wilf derived this by analyzing the prefix. If a composition starts with a large part, the remaining parts have fewer restrictions. If it starts with a small part, the remaining parts are heavily restricted to avoid completing a 123. Specifically, they establish a recurrence relation. By defining $f(n, k)$ as the number of such compositions of n with maximum part k, they sum over the possibilities for the first part c_1:

$$C(n) = C(n-1) + \sum_{k} (\text{restricted terms}).$$

□

6. Conclusion

Pattern avoidance is a crucial part of combinatorics. We have demonstrated that for permutations of length three, the avoidance condition is intimately tied to the Catalan numbers C_n. This relationship holds regardless of the specific pattern chosen, a fact we established through symmetries and recursive decomposition.

By visualizing these permutations through bijections to Dyck paths and stack operations, we gain a deeper understanding of their structure. The 132-avoiding permutations decompose into a "Left > Right" structure, while stack-sortable permutations naturally avoid 231.

These results have practical applications. Stack-sortable permutations are directly relevant to computer science, particularly in the design of sorting networks and the analysis of data streams. In bioinformatics, pattern avoidance concepts are used to analyze gene permutations and genome rearrangement distances.

References

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