Euler Circle Combinatorics Expository Paper (q Analogues)

Channing Yang

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Abstract

In this paper, we will explore an algebraic/number theoretic method of interpreting the partition function $p(n)$, which equals the number of ways to express n as a sum of (not necessarily distinct) positive integers (we will call each of these summands "parts"), where order does not matter. This seemingly unwieldy function has a surprisingly clean asymptotic formula found by Ramanujan and Hardy, namely $p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$. However, this asymptotic formula does not help in determining the number theoretic properties of $p(n)$; it is impossible to tell from it, for instance, when $p(n)$ is even. This makes Ramanujan's discovery that $p(5k+4) \equiv 0 \pmod 5$ for any nonnegative integer k extremely impressive. In this paper we will develop a generating function analogue for partitions to prove several results about partitions, ending with Ramanujan's celebrated $p(5k+4) \equiv 0 \pmod 5$ congruence.

1 q Analogues

We have the well-known generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

Indeed, writing the right hand side as

$$(q^{0\cdot 1} + q^{1\cdot 1} + q^{2\cdot 1} + \cdots)(q^{0\cdot 2} + q^{1\cdot 2} + q^{2\cdot 2} + \cdots)(q^{0\cdot 3} + q^{1\cdot 3} + q^{2\cdot 3} + \cdots)\cdots,$$

we see that each term in the product of the form $1+q^k+q^{2k}+\cdots$ corresponds to the number of k's in a partition. For example, the term

$$q^{2\cdot 1} \cdot q^{1\cdot 2} \cdot q^{0\cdot 3} \cdot q^{1\cdot 4} \cdot q^{0\cdot 5} \cdot q^{0\cdot 6} \cdots = q^8$$

in the product would correspond to the partition $1+1+2+4=8$.

It is natural to modify $p(n)$ slightly and derive similar generating functions, which we do below.

Definition 1.1 (Definition)**.** *We will denote $q_m(n)$ to be the number of partitions of n into at most m parts and $r_m(n)$ to be the number of partitions of n into parts that are at most m.*

Claim 1.2. *For all positive integers m and n, we have $q_m(n) = r_m(n)$.*

Proof. The claim is true for $m=1$ as we have $q_1(n) = r_1(n)$ for all n, so we will assume that $m \geq 2$.

We may use a similar argument as above to see that

$$\sum_{n=0}^{\infty} r_m(n) q^n = \prod_{k=1}^{m} \frac{1}{1-q^k}.$$

Therefore, it suffices to prove that

$$\sum_{n=0}^{\infty} q_m(n) q^n = \prod_{k=1}^{m} \frac{1}{1-q^k}.$$

We note that we may extend each partition of n into at most m positive parts to a partition of n into exactly m nonnegative parts by appending 0's. Therefore, we have

$$\sum_{n=0}^{\infty} q_m(n) q^n = \sum_{n_1 \geq n_2 \geq \cdots \geq n_m \geq 0} q^{n_1+n_2+\cdots+n_m}.$$

Definition 1.3 (Definition)**.** *Given variables $\lambda_1, \ldots, \lambda_k, q_1, \ldots, q_\ell$ and a series S where each term is of the form $\lambda_1^{e_1} \cdots \lambda_k^{e_k} q_1^{e_{k+1}} \cdots q_\ell^{e_{k+\ell}}$ for integers $e_1, \ldots, e_{k+\ell}$, we define $\underset{\geq}{\Omega}(S)$ to be the series obtained by removing all terms with negative exponents and setting all λ_i's equal to 1 for the other terms.*

For example,

$$\underset{\geq}{\Omega}(q_1 q_2 \lambda_1^{-1} \lambda_2 + q_1^2 \lambda_1 \lambda_2) = q_1^2.$$

Then, we may write

$$\sum_{n_1 \geq n_2 \geq \cdots \geq n_m \geq 0} q^{n_1+n_2+\cdots+n_m} = \underset{\geq}{\Omega} \sum_{n_1, n_2, \ldots, n_m \geq 0} q^{n_1+n_2+\cdots+n_m} \lambda_1^{n_1-n_2} \lambda_2^{n_2-n_3} \cdots \lambda_{m-1}^{n_{m-1}-n_m}.$$

The right hand side further equals

$$\underset{\geq}{\Omega} \sum_{n_1=0}^{\infty} (q\lambda_1)^{n_1} \sum_{n_2=0}^{\infty} \left(\frac{q\lambda_2}{\lambda_1}\right)^{n_2} \cdots \sum_{n_{m-1}=0}^{\infty} \left(\frac{q\lambda_{m-1}}{\lambda_{m-2}}\right)^{n_{m-1}} \sum_{n_m=0}^{\infty} \left(\frac{q}{\lambda_{m-1}}\right)^{n_m}.$$

Lemma 1.4. *We have*

$$\underset{\geq}{\Omega} \sum_{n=0}^{\infty}(\lambda x)^n \sum_{m=0}^{\infty}(y/\lambda)^m = \sum_{k=0}^{\infty} x^k \sum_{m=0}^{\infty}(xy)^m.$$

Proof. We have

$$\begin{aligned}
\underset{\geq}{\Omega} \left(\sum_{n=0}^{\infty}(\lambda x)^n\right)\left(\sum_{m=0}^{\infty}(y/\lambda)^m\right) &= \underset{\geq}{\Omega} \sum_{n=0}^{\infty}(\lambda x)^n \sum_{m=0}^{\infty}(y/\lambda)^m \\
&= \underset{\geq}{\Omega} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \lambda^{n-m}x^n y^m \\
&= \underset{\geq}{\Omega} \sum_{n=m}^{\infty}\sum_{m=0}^{\infty} x^n y^m \\
&= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} x^n y^m \\
&= \sum_{m=0}^{\infty}(xy)^m \sum_{k=0}^{\infty} x^k \\
&= \sum_{k=0}^{\infty} x^k \sum_{m=0}^{\infty}(xy)^m.
\end{aligned}$$

□

Now, we apply our Lemma with $x = q$, $y = q\lambda_2$, and $\lambda = \lambda_1$ to write

$$\underset{\geq}{\Omega} \sum_{n_1=0}^{\infty}(q\lambda_1)^{n_1} \sum_{n_2=0}^{\infty}\left(\frac{q\lambda_2}{\lambda_1}\right)^{n_2} = \sum_{k=0}^{\infty} q^k \sum_{m=0}^{\infty}(q^2\lambda_2)^m.$$

This would remove all terms with a negative exponent on λ_1 and set $\lambda_1 = 1$ for all other terms. We apply our Lemma again with $x = q^2$, $y = q\lambda_3$, and $\lambda = \lambda_2$ to write

$$\begin{aligned}
\underset{\geq}{\Omega} \sum_{k=0}^{\infty} q^k \sum_{m=0}^{\infty}(q^2\lambda_2)^m \sum_{n_3=0}^{\infty}\left(\frac{q\lambda_3}{\lambda_2}\right)^{n_3} &= \left(\sum_{k=0}^{\infty} q^k\right)\left(\underset{\geq}{\Omega} \sum_{m=0}^{\infty}(q^2\lambda_2)^m \sum_{n_3=0}^{\infty}\left(\frac{q\lambda_3}{\lambda_2}\right)^{n_3}\right) \\
&= \sum_{k=0}^{\infty} q^k \sum_{\ell=0}^{\infty} q^{2\ell} \sum_{m=0}^{\infty}(q^3\lambda_3)^m.
\end{aligned}$$

This would remove all terms with a negative exponent on λ_2 and set $\lambda_2 = 1$ for all other terms. We may continue doing this to find that

$$\underset{\geq}{\Omega} \sum_{n_1=0}^{\infty} (q\lambda_1)^{n_1} \sum_{n_2=0}^{\infty} \left(\frac{q\lambda_2}{\lambda_1}\right)^{n_2} \cdots \sum_{n_{m-1}=0}^{\infty} \left(\frac{q\lambda_{m-1}}{\lambda_{m-2}}\right) \sum_{n_m=0}^{\infty} \left(\frac{q}{\lambda_{m-1}}\right)^{n_m}$$

equals

$$\sum_{k_1=0}^{\infty} q^{k_1} \sum_{k_2=0}^{\infty} q^{2k_2} \cdots \sum_{k_m=0}^{\infty} q^{mk_m}.$$

We have

$$\sum_{k_1=0}^{\infty} q^{k_1} \sum_{k_2=0}^{\infty} q^{2k_2} \cdots \sum_{k_m=0}^{\infty} q^{mk_m} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

This proves that $q_m(n) = r_m(n)$ for all positive integers m and n. □

Having seen the power of partition generating functions in turning combinatorial problems into algebraic ones, we now dedicate the remainder of the paper to proving the number theoretic $p(5k+4) \equiv 0 \pmod 5$. We will need to prove several intermediate theorems along the way. [Definition] Fix two positive integers m and n, and define the generating function

$$R_q(m,n) = \sum_{n \geq k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} q^{k_1+k_2+\cdots+k_m}.$$

Claim 1.5. *For all positive integers $m, n \geq 2$, we have*

$$R_q(m,n) = R_q(m,n-1) + q^n R_q(m-1,n).$$

Proof. We have

$$\begin{aligned}
R_q(m,n) &= \sum_{n \geq k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} q^{k_1+k_2+\cdots+k_m} \\
&= \sum_{n > k_1 \geq k_2 \geq \cdots \geq k_m \geq 0} q^{k_1+k_2+\cdots+k_m} + \sum_{n = k_1 \geq k_2 \geq \cdots \geq k_m} q^{k_1+k_2+\cdots+k_m} \\
&= \sum_{n-1 \geq k_1 \geq k_2 \geq \cdots \geq k_m} q^{k_1+k_2+\cdots+k_m} + q^n \sum_{n \geq k_2 \geq \cdots \geq k_m \geq 0} q^{k_2+\cdots+k_m} \\
&= R_q(m,n-1) + q^n R_q(m-1,n).
\end{aligned}$$

□

[Definition] For all positive integers n, we will denote

$$(n)_q = R_q(n-1,1) = 1 + q + \cdots + q^{n-1}.$$

We will further denote

$$(n)!_q = (n)_q(n-1)_q \cdots (1)_q.$$

Claim 1.6. *For all positive integers m and n, we have*

$$R_q(m,n) = \frac{(m+n)!_q}{(m)!_q(n)!_q}.$$

Proof. The result is true for $m = 1$ as

$$R_q(1,n) = \frac{(n+1)!_q}{(n)!_q(1)!_q} = 1 + q + \cdots + q^n.$$

Similarly, the result is true for $n = 1$ as

$$R_q(m,1) = \frac{(m+1)!_q}{(m)!_q(1)!_q} = 1 + q + \cdots + q^m.$$

We now prove the result for general m and n by induction on $m+n$, with the base cases of $m+n = 2, 3$ already covered above. We now assume that, for some positive integer k, the result holds for all $m+n = k$, and we will prove that the result holds for all $m+n = k+1$. If $m = 1$ or $n = 1$, then we may use the argument above. We will now assume that $m, n \geq 2$. We apply the inductive hypothesis to find that

$$\begin{aligned}
R_q(m,n) &= R_q(m,n-1) + q^n R_q(m-1,n) \\
&= \frac{(m+n-1)!_q}{(m)!_q(n-1)!_q} + \frac{(m+n-1)!_q}{(m-1)!_q(n)!_q} \\
&= \frac{(m+n-1)!_q(q^n(m)_q + (n)_q)}{(m)!_q(n)!_q} \\
&= \frac{(m+n-1)!_q(q^n(1+q+\cdots+q^{m-1}) + (1+q+\cdots+q^{n-1}))}{(m)!_q(n)!_q} \\
&= \frac{(m+n)!_q}{(m)!_q(n)!_q}.
\end{aligned}$$

This completes the inductive step. ∎

Note that we may write

$$R_q(m,n) = \frac{(1-q^{m+n})\cdots(1-q)}{(1-q^m)\cdots(1-q)(1-q^n)\cdots(1-q)}.$$

We now prove an important theorem.

Theorem 1.7. *(Jacobi Triple Product Identity) We have*

$$\prod_{m=1}^{\infty}(1-q^{2m})(1+xq^{2m-1})(1-x^{-1}q^{2m-1}) = \sum_{k=-\infty}^{\infty} q^{k^2}x^k,$$

Proof. We first observe that

$$(1+xq)(1+xq^2)\cdots(1+xq^n) = \sum_{k=0}^{n}\left(\sum_{n\geq x_1>x_2>\cdots>x_k>0} q^{x_1+x_2+\cdots+x_k}\right)x^k.$$

We note that the right hand side equals

$$\sum_{k=0}^{n}\left(\sum_{n-k\geq x_1-k\geq x_2-(k-1)\geq\cdots\geq x_k-1\geq 0} q^{(x_1-k)+(x_2-(k-1))+\cdots+(x_k-1)}q^{\frac{k(k+1)}{2}}\right)x^k.$$

By our previous definition of $R_q(m,n)$, this further equals

$$\sum_{k=0}^{n} R_q(k,n-k)q^{\frac{k(k+1)}{2}}x^k.$$

We now make the substitution $x \to -\frac{x}{q}$ to obtain

$$(1-x)(1-xq)\cdots(1-xq^{n-1}) = \sum_{k=0}^{n} R_q(k,n-k)q^{\frac{k(k-1)}{2}}(-x)^k.$$

We further make the substitution $n \to 2n$ to obtain

$$(1-x)(1-xq)\cdots(1-xq^{2n-1}) = \sum_{k=0}^{n} R_q(k,2n-k)q^{\frac{k(k-1)}{2}}(-x)^k.$$

We may rewrite the left hand side as

$$(-x)^n q^{\frac{n(n-1)}{2}}(1-x^{-1})(1-x^{-1}q^{-1})\cdots(1-x^{-1}q^{-n+1})(1-xq^n)\cdots(1-xq^{2n-1}).$$

We make the substitution $x \to \frac{x}{q^n}$ to obtain

$$\left(-\frac{x}{q^n}\right)^n q^{\frac{n(n-1)}{2}}(1-x^{-1}q)(1-x^{-1}q^2)\cdots(1-x^{-1}q^n)(1-x)(1-xq)\cdots(1-xq^{n-1})$$

equals

$$\sum_{k=0}^{n} R_q(k, 2n-k)q^{\frac{k(k-1)}{2}}\left(-\frac{x}{q^n}\right)^k.$$

We multiply both of these expressions by $q^{\frac{n(n+1)}{2}}(-x)^{-n}$ to obtain

$$(1-x^{-1}q)\cdots(1-x^{-1}q^n)(1-x)\cdots(1-xq^{n-1}) = \sum_{k=0}^{2n} R_q(k, 2n-k)q^{\frac{(n-k)(n-k+1)}{2}}(-x)^{-n+k}.$$

We make the substitution $k \to -n+k$, so that the right hand side becomes

$$\sum_{k=-n}^{n} R_q(n+k, n-k)q^{\frac{k(k-1)}{2}}(-x)^k.$$

Now, note that fixing k and letting n tend toward infinity, we have

$$\lim_{n\to\infty} R_q(n+k, n-k) = \lim_{n\to\infty} \frac{(1-q^{2n})\cdots(1-q)}{(1-q^{n+k})\cdots(1-q)(1-q^{n-k})\cdots(1-q)} = \prod_{m=1}^{\infty}\frac{1}{1-q^m},$$

as the numerator tends to $\prod_{m=1}^{\infty}(1-q^m)$ and the denominator tends to $\left(\prod_{m=1}^{\infty}(1-q^m)\right)^2$. Therefore, letting n tend to infinity on both sides of the equation

$$(1-x^{-1}q)\cdots(1-x^{-1}q^n)(1-x)\cdots(1-xq^{n-1}) = \sum_{k=-n}^{n} R_q(n+k, n-k)q^{\frac{k(k-1)}{2}}(-x)^k,$$

we have

$$\prod_{m=1}^{\infty}(1-x^{-1}q^m)(1-xq^{m-1}) = \prod_{m=1}^{\infty}\frac{1}{1-q^m}\sum_{k=-\infty}^{\infty} q^{\frac{k(k-1)}{2}}(-x)^k.$$

Therefore, we have

$$\prod_{m=1}^{\infty}(1-x^{-1}q^m)(1-xq^{m-1})(1-q^m) = \sum_{k=-\infty}^{\infty} q^{\frac{k(k-1)}{2}}(-x)^k.$$

We now substitute $q \to q^2$ and $x \to -xq$ to get that

$$\prod_{m=1}^{\infty}(1-q^{2m})(1+xq^{2m-1})(1-x^{-1}q^{2m-1}) = \sum_{k=-\infty}^{\infty} q^{k^2}x^k,$$

as desired. □

[Definition] We will define the following.

- $(a)_n = (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$.
- $(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$.

We now introduce an intermediate theorem.

Theorem 1.8. *(Jacobi's Identity) We have*

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}} = (q;q)_\infty^3.$$

Proof. We make the substitution $x \to x^2q$ in the Jacobi Triple Product Identity to obtain

$$\begin{aligned}\sum_{k=-\infty}^{\infty} q^{k^2+k}x^{2k} &= (-x^2q^2;q^2)_\infty\left(-\frac{1}{x^2};q^2\right)_\infty(q^2;q^2)_\infty \\ &= \left(1+\frac{1}{x^2}\right)(-x^2q^2;q^2)_\infty\left(-\frac{q^2}{x^2};q^2\right)_\infty(q^2;q^2)_\infty.\end{aligned}$$

This implies that

$$\frac{1}{x+\frac{1}{x}}\sum_{k=-\infty}^{\infty}x^{2k+1}q^{k^2+k} = (-x^2q^2;q^2)_\infty\left(-\frac{q^2}{x^2};q^2\right)_\infty(q^2;q^2)_\infty.$$

We now take the limit as x tends towards i (the imaginary unit) of both sides of this equation. Note that $i+\frac{1}{i}=0$ and

$$\sum_{k=-\infty}^{\infty} i^{2k+1}q^{k^2+k} = \sum_{k=0}^{\infty}(i^{2k+1}+i^{2(-k-1)+1})q^{k^2+k} = 0.$$

Therefore, we may apply L'Hopital's Rule to obtain

$$\lim_{x\to i} \quad \frac{1}{x+\frac{1}{x}} \sum_{k=-\infty}^{\infty} x^{2k+1} q^{k^2+k} = \frac{1}{2} \sum_{k=-\infty}^{\infty} (2k+1)(-1)^k q^{k^2+k}.$$

$$= \frac{1}{2} \left(\sum_{k=0}^{\infty} ((2k+1)(-1)^k + (2(-k-1)+1)(-1)^{-k-1}) q^k \right)$$

$$= \sum_{k=0}^{\infty} (2k+1)(-1)^k q^{k^2+k}.$$

On the other hand, we also have

$$\lim_{x\to i} \quad (-x^2q^2;q^2)_\infty \left(-\frac{q^2}{x^2};q^2\right)_\infty (q^2;q^2)_\infty = (q^2;q^2)_\infty^3.$$

This proves Jacobi's Identity. □

We are now ready for our final interemediate theorem.

Theorem 1.9. *(Euler's Pentagonal Number Theorem) We have*

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k-1)}{2}} = (q;q)_\infty.$$

Proof. We note that

$$(q;q)_\infty = (1-q)(1-q^2)\cdots = \sum_{k_1,k_2,\cdots\in\{0,1\}} q^{k_1+2k_2+\cdots}(-1)^{k_1+k_2+\cdots}.$$

Therefore, the coefficient of q^n in $(q;q)_\infty$ is equal to the number of partitions of n into an even number of distinct parts minus the number of partitions of n into an odd number of distinct parts. We will define an involution on the set of partitions of a positive integer n to easily compute the difference. First, let us make the following definition. [Definition] Given a partition $P = p_1 + p_2 + \cdots + p_m$ into parts $p_1 > p_2 > \cdots > p_m$, define $f(P)$ be the largest index i so that $p_i = p_1 - i + 1$ and $g(P)$ to be the value of p_m. For example, for the partition $P = 3+2+1$ of 6, we have $f(P) = 3$ and $g(P) = 1$. We are now ready to define the involution.

- If a partition $P = p_1 + p_2 + \cdots + p_m$ into parts $p_1 > p_2 > \cdots > p_m$ satisfies $f(P) \geq g(P)$, then we map P to the partition

  $$P' = (p_1 + 1) + (p_2 + 1) + \cdots + (p_{p_m} + 1) + p_{p_m+1} + \cdots + p_{m-1}.$$

  For example, we map the partition $P = 7 + 6 + 5 + 4 + 2$ of 24 to the partition $P' = (7 + 1) + (6 + 1) + 5 + 4 = 8 + 7 + 5 + 4$. Note that $f(P') = p_m < p_{m-1} = g(P')$. Furthermore, P' has one less part than P.

- If a partition $P = p_1 + p_2 + \cdots + p_m$ into parts $p_1 > p_2 > \cdots > p_m$ satisfies $g(P) > f(P)$, then we set $f(P) = t$ and map P to the partition

  $$P' = (p_1 - 1) + (p_2 - 1) + \cdots + (p_t - 1) + p_{t-1} + \cdots + p_m + t.$$

  For example, we map the partition $P = 9 + 8 + 4 + 3$ of 24 to the partition $P' = (9 - 1) + (8 - 1) + 4 + 3 + 2 = 8 + 7 + 4 + 3 + 2$. Note that $f(P') \geq t = g(P')$. Furthermore, P' has one more part than P.

Note that this involution is not well defined for some partitions P. Indeed, all partitions P with $f(P) \geq g(P)$ and $g(P) \leq m-1$ are mapped to a unique partition into distinct parts, but when $g(P) = m$,

$$P' = (p_1 + 1) + (p_2 + 1) + \cdots + (p_{p_m} + 1) + p_{p_m+1} + \cdots + p_{m-1}$$

is not well-defined. As $f(P) \leq m$, the only partitions P satisfying $f(P) \geq g(P)$ that do not get mapped to a unique partition into distinct parts are those with $f(P) = g(P) = m$. Similarly, all partitions P with $g(P) > f(P)$ and $f(P) \leq m - 1$ are mapped to a unique partition into distinct parts as $p_t - 1 > p_{t-1}$ by the definition of $f(P)$ and $p_m > t$ by assumption. However, if $f(P) = m$, then P is mapped to a unique partition into distinct parts only if $p_m - 1 > t$. This means that the only partitions P satisfying $g(P) > f(P)$ that do not get mapped to a unique partition into distinct parts are those with $f(P) = m$ and $g(P) = m + 1$. For the partitions P whose image is a unique partition into distinct parts, however, it is verifiable that the image of P's image is P. Therefore, all that remains is to characterize the n for which the exceptions of partitions described above may occur. The partitions P into m distinct parts satisfying $f(P) = g(P) = m$ are of the form $(2m-1)+(2m-2)+\cdots+m$, and the partitions P satisfying $f(P) = m$ and $g(P) = m + 1$ are of the form $(2m) + (2m - 1) + \cdots + (m + 1)$. Such partitions occut only for n of the form $\frac{m(3m-1)}{2}$ or $\frac{m(3m+1)}{2} = \frac{-m(-3m-1)}{2}$. Therefore, the coefficient of q^n in $(q;q)_\infty$ is 0 unless n equals $\frac{m(3m-1)}{2}$ for some integer m, in which case it is $(-1)^m$. This proves Euler's Pentagonal Number Theorem. □

Note that $\frac{j(3j-1)}{2} = \frac{(-j)(-3j+1)}{2}$, so we may rewrite Euler's Pentagonal Number Theorem as

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k+1)}{2}} = (q;q)_\infty.$$

We are finally ready to put everything together and prove that $p(5k+4) \equiv 0 \pmod 5$ for all nonnegative integers k. In what follows, we will say that two power series are congruent mod 5 if, for any $n \geq 0$, the coefficients of q^n in both power series are congruent mod 5. Let us consider the generating function

$$(q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)q^{m+1} = (1 - q^5 - q^{10} + q^{25} + \cdots) \sum_{m=0}^{\infty} p(m)q^{m+1}.$$

Note that the coefficient of q^{5n+5} in this generating function is

$$p(5n+4) + (a_1 p(5n-1) + a_2 p(5n-6) + \cdots),$$

where $a_1, a_2, \ldots$ are integers. Therefore, if we are able to prove that the coefficient of q^{5n+5} in this generating function above is always a multiple of 5, then it will follow by induction that $p(5n+4) \equiv 0 \pmod 5$ for all $n \geq 0$ (for the base case, we have $p(4) = 5$). We may write

$$\begin{aligned}(q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)q^{m+1} &= \frac{q(q^5;q^5)_\infty}{(q;q)_\infty} \\ &= q(q;q)_\infty^4 \frac{(q^5;q^5)_\infty}{(q;q)_\infty^5}.\end{aligned}$$

We have

$$\frac{1}{(q;q)_\infty^5} = \prod_{k=1}^{\infty} (1 + q^k + q^{2k} + \cdots)^5.$$

Taking the coefficients mod 5, we have

$$\prod_{k=1}^{\infty} (1 + q^k + q^{2k} + \cdots)^5 \equiv \prod_{k=1}^{\infty} (1 + q^{5k} + q^{10k} + \cdots) \pmod 5.$$

Therefore,

$$\frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} \equiv \prod_{k=1}^{\infty} (1 - q^{5k})(1 + q^{5k} + q^{10k} + \cdots) \equiv 1 \pmod 5.$$

We now have

$$\begin{aligned}
q(q;q)_\infty^4\frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} &\equiv q(q;q)_\infty^4 \pmod 5\\
&\equiv q(q;q)_\infty^3(q;q)_\infty \pmod 5\\
&\equiv q\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{\frac{k(k+1)}{2}}\right)\left(\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{j(3j+1)}{2}}\right) \pmod 5\\
&\equiv \sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}(2k+1)q^{1+\frac{k(k+1)}{2}+\frac{j(3j+1)}{2}} \pmod 5.
\end{aligned}$$

We claim that all integers j and k for which

$$1+\frac{k(k+1)}{2}+\frac{j(3j+1)}{2}\equiv 0 \pmod 5$$

satisfy $2k+1\equiv 0 \pmod 5$, which would prove that the coefficient of q^{5n+5} is a multiple of 5 for all $n\geq 0$. We note that

$$2(j+1)^2+(2k+1)^2 = 8\left(1+\frac{j(3j+1)}{2}+\frac{k(k+1)}{2}\right)-10j^2-5.$$

Therefore, if

$$1+\frac{k(k+1)}{2}+\frac{j(3j+1)}{2}\equiv 0 \pmod 5,$$

then we also have

$$2(j+1)^2+(2k+1)^2\equiv 0 \pmod 5.$$

As $\left(\frac{-2}{5}\right)=-1$, this implies that $j+1\equiv 2k+1\equiv 0 \pmod 5$, and we are done.

References

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