# Winning Probabilities in Matroid Bingo

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**Abstract**

In this paper, we define the matroid using the game of matroid bingo to give intuition for the axioms of matroids, as well as explore an equation for the probabilities of winning matroid bingo cards.

# 1 Matroids

## 1.1 Bingo

We begin by first considering the game of matroid bingo. This is a game in which players have cards with a subset of $[n]$ written on them, and a caller reads randomly selected positive integers in $[n]$. A player circles an integer on their card if it is called, and wins when they circle every number on their card. Additionally, the cards are made with a few special properties:

1. There are no empty cards; no one wins before the game begins.
2. Every card has a winning chance, no card is a proper subset of another.
3. There cannot be a tie; the numbers on each card are chosen to prevent a tie.

An example collection of matroid bingo cards, for 5 players playing with the integers from 1 to 8 would be

$$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{5,6,7,8\}.$$

Though this is a game even simpler than bingo, as it does not have the feature of winning by only having a specific subset of the card called, it can still be an interesting game to play. This is not always the case, as

$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$$

is also a valid set of cards for five players using the positive integers from 1 to 5, but there are many more interesting matroid bingo card sets. Now, an example of an invalid card set would be

$$\{1, 2\}, \{1, 3\}$$

as a tie occurs if 1 is called after 2 and 3.

More importantly, these properties of the cards actually encode the properties of a definition of a matroid. A more formal definition is as follows:

**Definition 1.1.** A **matroid** $M = (E, \mathcal{C})$ consists of a non-empty finite set $E$, and a collection $\mathcal{C}$ of non-empty subsets of $E$ called *circuits* satisfying the following properties:

1. No circuit properly contains another circuit.
2. If $C_1$ and $C_2$ are distinct circuits containing an element $e$, then there exists a circuit that is a subset of $C_1 \cup C_2$ that does not contain $e$.

These are not just similar to the axioms of matroid bingo; they are almost exactly the same. The first axiom for matroid bingo, that no card is empty, is given in the definition of $\mathcal{C}$, as it is a collection of *non-empty* subsets of $E$. The second axiom, that no card is a proper subset of another, corresponds to the first axiom for the circuit definition of a matroid. The third axiom, that there cannot be a tie, corresponds to the second axiom of the formal definition. For two cards to tie, they must have all their numbers chosen except some $e$. The second axiom forces there to be a circuit that contains a proper subset of the numbers on both cards, excluding $e$. As the two cards would tie if $e$ were called, all the numbers on the third card must have already been called, so the game would already be over.

There are plenty of other equivalent definitions of matroids, called *cryptomorphisms*, which can be found in [Wil73], but this is the one we'll focus on.

## 1.2 Monotonicity violations

Now that we are considering a game, that of matroid bingo, the natural follow-up would be to consider a winning strategy. For instance, consider once again the example of

$$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{5,6,7,8\}.$$

Out of these cards, which one would be the best choice to win?

As it turns out, this is a rather unusual case, in which although there are shorter bingo cards that would require fewer numbers called to win, and so naturally would seem to have a better chance of victory, a longer bingo card has a higher chance of winning, that being $\{5, 6, 7, 8\}$. In [BDFG25] this unusual occurrence is called a **monotonicity violation**. The exact winning odds for this card set are as follows:

| Card | Probability |
|---|---|
| $\{1,2,3\}$ | $11/56 \approx 0.1964$ |
| $\{1,2,4\}$ | $11/56 \approx 0.1964$ |
| $\{1,3,4\}$ | $11/56 \approx 0.1964$ |
| $\{2,3,4\}$ | $11/56 \approx 0.1964$ |
| $\{5,6,7,8\}$ | $3/14 \approx 0.2143$ |

# 2 Winning at bingo

At this point we intend to find the actual probabilities of winning. Given a set of legal bingo cards $\mathcal{C}$, or equivalently the circuits of a matroid $\mathcal{M}$, we denote by $P(C)$ the probability that a given circuit $C$ wins the game. From our definitions, $P(C) > 0$ for all circuits $C$, and

$$\sum_{C \in \mathcal{C}} P(C) = 1.$$

We then make an important definition.

**Definition 2.1.** For a matroid $\mathcal{M}$, we define the **rank** of the matroid $r(\mathcal{M})$ to be the size of the largest subset not containing any circuit.

Now we can go through the formula and proof for probability of circuits found in [BDFG25].

**Theorem 2.2.** *Given a matroid $\mathcal{M}$ on $E = [n]$ and a circuit $C$ of $\mathcal{M}$, we have*

$$P(C) = \sum_{t=|C|}^{r(\mathcal{M})+1} P(C)_t$$

*where*

$$P(C)_t = \frac{|C|}{n} \cdot |\mathcal{I}_{C,t-|C|}| \cdot \binom{n-1}{t-1}^{-1}$$

*and $\mathcal{I}_{C,k}$ denotes the collection of all $k$-element subsets $S$ of $E \setminus C$ where $C$ is the only circuit contained in $S \cup C$.*

*Proof.* We will prove the equivalent statement that if $P_C(t)$ denotes the set of permutations of $E$ in which a given circuit $C$ is the first one to be completed, and this happens in round $t$, then

$$|P_C(t)| = |C| \cdot |\mathcal{I}_{C,t-|C|}| \cdot (t-1)! \cdot (n-t)!.$$

We can count all such permutations by

1. Choosing an element $e$ to be the last element of $C$ called (in round $t$).
2. Choosing a $(t - |C|)$-element subset $S \subseteq E \setminus C$ of additional elements called out before $C$ wins, or in other words, an element of $\mathcal{I}_{C,t-|C|}$.
3. Choosing an ordering of $(C \cup S) \setminus \{e\}$.
4. Choosing an ordering of $E \setminus (C \cup S)$.

This gives us our desired formula,

$$|P_C(t)| = |C| \cdot |\mathcal{I}_{C,t-|C|}| \cdot (t-1)! \cdot (n-t)!.$$

And when we divide by the $n!$ total number of permutations of $E$, we get

$$P(C)_t = \frac{|P_C(t)|}{n!} = \frac{|C|}{n} \cdot |\mathcal{I}_{C,t-|C|}| \cdot \binom{n-1}{t-1}^{-1},$$

and since $P(C)_t$ is the probability of the circuit $C$ winning on round $t$, the sum over $t$ up to $r(\mathcal{M}) + 1$, the number of elements that need to be chosen to guarantee a winner, gives us the probability of $C$ winning. □

The most unclear part of this proof is the $\mathcal{I}_{C,t-|C|}$, so we'll take a closer look at it. It is defined as the collection of all $(t - |C|)$-element subsets $S$ of $E \setminus C$ where $C$ is the only circuit contained in $S \cup C$. In other words, it is the collection of sets $S$ of size $k$ where, when added to the bingo card $C$, the only circuit in the resulting set is still that of $C$. This ensures our set of numbers to be called will not have another circuit that wins before $C$.

## References


[BDFG25] Matthew Baker, Hope Dobbelaere, Brennan Fullmer, and Patrik Gajdoš. Matroid bingo, 2025.

[Wil73] R. J. Wilson. An introduction to matroid theory. *Am. Math. Mon.*, 80:500–525, 1973.