COMBINATORIAL DESIGNS

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Abstract

In this paper, we explore well-known combinatorial designs, focusing on balanced incomplete block designs (BIBDs). First we define BIBDs with the example of the Fano plane, and prove some basic properties relating their parameters. Next, we employ linear algebra techniques on the incidence matrix of a BIBD to prove the famous Fisher's inequality. Finally, we discuss finite projective planes and how they relate to BIBDs.

1. Balanced Incomplete Block Designs

Definition 1.1. A *design* is defined as a pair $(V, \mathcal{B})$, where V is a finite set of elements, called *points*, and $\mathcal{B}$ is a collection of nonempty subsets of V, called *blocks*.

Note that $\mathcal{B}$ is a multiset, so it is allowed to have repeated blocks. However, we are mainly interested in designs without repeated blocks, which are called *simple designs*. Some interesting designs have symmetric properties based on the sizes of the blocks and the points shared between blocks. We give them a special name:

Definition 1.2. For $t \in \mathbb{N}$, a *t-design* is a design $(V, \mathcal{B})$ such that each block in $\mathcal{B}$ has the same number of points, and there is a constant λ_t such that any t-tuple of points in V can be found in exactly λ_t blocks in $\mathcal{B}$.

We will look closer at the balanced incomplete block designs (BIBDs), which are designs that have a few properties satisfied that give them additional structure. We specify the type of BIBD based on five parameters: v, b, r, k, and λ.

Definition 1.3. For positive integers v, b, r, k, λ with $2 \leq k < v$, a (v, b, r, k, λ)-*BIBD* is a design $(V, \mathcal{B})$ that satisfies the following properties:

(1) there are v points,
(2) there are b blocks
(3) each point is in exactly r blocks
(4) each block contains exactly k points,
(5) each pair of distinct points is in exactly λ blocks.

We call these designs "balanced" in the sense that each pair of points occurs the same number of times, and "incomplete" because $k < v$, so the blocks cannot contain the entire set of points. Additionally, property (3) and property (5) are equivalent to the design being a 1-design and a 2-design, respectively. Properties (1) and (2) are made true by definition, and we will show later that we can uniquely determine k from the rest of the information, so an alternative definition for a BIBD is a design that is both a 1-design and a 2-design.

To illustrate the idea of a BIBD, we will consider a well-known example.

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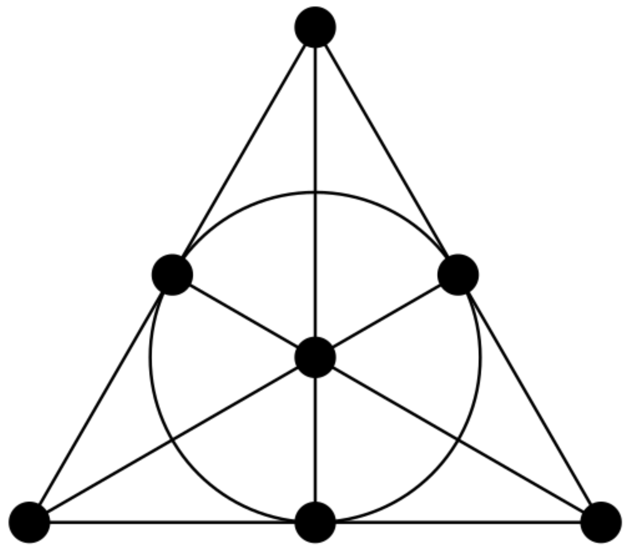

Figure 1. The Fano plane

Example. Suppose we have a tournament of 7 teams, and we want each team to play against each other. However, the game we are playing requires three teams to play at a time, after which a ranking of the three teams is determined. Due to this, we cannot use a typical round-robin style tournament, so we want to know if it is possible to arrange such a tournament without requiring a team to play another team on more than one occasion.

We can represent this scenario with a BIBD, where the points represent teams and the blocks represent matches. In particular, for this BIBD we have $v = 7$ because there are 7 teams, $k = 3$ because there are 3 teams in each match, and $\lambda = 1$ because each pair of teams plays each other exactly once. If a BIBD with these conditions exists, then the tournament scenario would be possible.

It turns out that this is possible, and if we label the teams from 0 to 6, then a possible construction for the blocks is as follows:

$$\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}$$

This design has $b = 7$ blocks, and each point appears in exactly $r = 3$ blocks, so it is a BIBD.

This BIBD is known as the Fano plane, and it is the simplest nontrivial example of a BIBD. It is often represented as the diagram in Figure 1. In this diagram, the 7 vertices represent the points, and the 7 lines represent the blocks. Since it is a BIBD, it has the properties that there is a fixed number of lines on each vertex, and there is a fixed number of vertices on each line. Furthermore, each pair of lines intersects at exactly one vertex.

We can use combinatorial double-counting arguments to relate the parameters of a BIBD, which gives us some necessary conditions for the existence of a BIBD for certain parameters.

Proposition 1.4. *For any BIBD, the following equations hold:*

$$vr = bk \tag{1.1}$$

$$r(k-1) = \lambda(v-1). \tag{1.2}$$

Proof. To prove equation (1.1), we will count in two ways the number of ways to choose a block and a point in the block, that is, the cardinality of the set

$$\{(B, x) : B \in \mathcal{B}, x \in B\}.$$

The cardinality of this set can be counted either by iterating over the points or the blocks of the design. There are v points, and each point is contained in r blocks, so the number of such pairs is vr. Alternatively, there are b blocks, and each block contains k points, so there are bk elements, meaning that $vr = bk$.

Next, for equation (1.2) we will fix a point a and count the number of ways to choose a block containing a, then choose a point in the block other than a. The corresponding set is

$$\{(B, x) : B \in \mathcal{B}, a \in B, x \in B, x \neq a\}.$$

Firstly, there are r blocks in B containing a, and for each such block there are $k-1$ choices for x that are not equal to a, so the cardinality of the set is $r(k-1)$. On the other hand, there are $v-1$ choices for x since $x \neq a$, and for each x there are λ blocks in B containing both a and x, so the cardinality of the set is $\lambda(v-1)$. Therefore, $r(k-1) = \lambda(v-1)$. ■

Using these two equations, we can derive all five parameters from just knowing the values of three of them. Due to this, we often just specify a design as a (v, k, λ)-design, and we call these three parameters the *primary parameters*, while b and r are the *secondary parameters*. Then, using the equations we just found, it is easy to see that $r = \frac{\lambda(v-1)}{k-1}$ and $b = \frac{vr}{k} = \frac{\lambda(v^2-v)}{k^2-k}$.

2. Fisher's Inequality

It is often helpful to represent a design in the form of a matrix, so that we can use linear algebra tools to perform operations on the entries. Using this method, we will eventually prove Fisher's inequality, which is considered the most fundamental theorem for BIBDs.

Definition 2.1. For a (v, k, λ)-design $(V, \mathcal{B})$ with points V_1 through V_v and blocks $\mathcal{B}_1$ through $\mathcal{B}_b$, we define its *incidence matrix* as follows: the $v \times b$ matrix A such that $A_{i,j} = 1$ if $V_i \in \mathcal{B}_j$, and $A_{i,j} = 0$ otherwise.

Example. The incidence matrix for the Fano plane is

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

This matrix is quite useful because it clearly encodes a lot of the important information about the design. In particular, if A is the incidence matrix of a (v, b, r, k, λ) design, then each row has exactly r ones since each point is in r blocks, each column has exactly k ones since each block contains k points, and the dot product of any two distinct rows is λ since each pair of points is found in exactly λ blocks. To utilize this property, it is useful to consider the $v \times v$ matrix formed by multiplying A by its transpose: AA^T.

Lemma 2.2. *The matrix AA^T has all diagonal entries equal to r, and the rest of the entries are equal to λ.*

Proof. The (i, j) entry of this matrix is the dot product of the ith and jth rows of A. On the diagonal, $i = j$, so the entry in AA^T is the dot product of the ith row with itself. This counts the number of ones in the row, which is r since each point is in r blocks. For all other entries, we showed earlier that the dot product of any two distinct rows is λ, so the remaining entries of AA^T are all filled with λ. ■

We are interested in the rank of the matrix AA^T, since this will give us our desired bound. To find this, we will use the fact that the rank of a matrix is preserved when we perform elementary row and column operations. If we can reduce AA^T to a upper-triangular matrix with no zero rows, then we will know that the rows are all linearly independent, so the rank is v.

Lemma 2.3. *The rank of AA^T is equal to v.*

Proof. From 2.2, we begin with the following $v \times v$ matrix:

$$AA^T = \begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \lambda & \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \lambda & \cdots & r \end{bmatrix}.$$

By subtracting the first row from each of the others, we get

$$\begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda - r & r - \lambda & 0 & \cdots & 0 \\ \lambda - r & 0 & r - \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda - r & 0 & 0 & \cdots & r - \lambda \end{bmatrix}.$$

Next, we can cancel out the remaining nonzero entries in the first column by adding to the first row each of the other columns:

$$\begin{bmatrix} r + (v-1)\lambda & \lambda & \lambda & \cdots & \lambda \\ 0 & r - \lambda & 0 & \cdots & 0 \\ 0 & 0 & r - \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r - \lambda \end{bmatrix}.$$

Since this matrix is upper-triangular and all the diagonal entries are nonzero, the rank of the matrix is v, so the original matrix also has $\text{rank}(AA^T) = v$. ■

Now that we have found the exact value of $\text{rank}(AA^T)$, we can use another property from linear algebra to put a bound on $\text{rank}(AA^T)$. This will prove our theorem.

Theorem 2.4 (Fisher's Inequality)**.** *For any balanced incomplete block design, the number of blocks is at least the number of points, that is,*

$$b \geq v.$$

Proof. We have shown in 2.3 that $\text{rank}(AA^T)$ is equal to v. Additionally, we know from linear algebra that the rank of a product of two matrices is at most the rank of each of the matrices. Matrices A and A^T have v and b columns, respectively. Thus, $\text{rank}(AA^T) = v$ cannot exceed b, so we have shown that

$$b \geq v$$

for all BIBDs. ■

3. Finite Projective Planes

The Fano plane from earlier is a special example of a BIBD because it satisfies the property that $v = b$, the number of points is equal to the number of blocks. This is the equality case of Fisher's inequality that we proved in the previous section. We give BIBDs of this form a special name.

Definition 3.1. A BIBD is a *symmetric balanced incomplete block design (SBIBD)* if it satisfies the condition that $v = b$.

Using our first equation from 1.4, we get as a corollary of this condition that $r = k$, so the number of blocks in which each point appears is equal to the number of points in each block.

We can generalize the idea of the Fano plane into designs with more points and blocks. The Fano plane is part of a special class of SBIBDs called finite projective planes, which are typically defined as follows.

Definition 3.2. A *finite projective plane of order* n is defined as a set of points and lines that satisfy the following properties:

(1) Each line goes through exactly $n + 1$.
(2) Each point lies on exactly $n + 1$ lines.
(3) For each pair of distinct points, there is exactly 1 line that goes through both of them.
(4) Each pair of distinct lines intersects at exactly 1 point.

As we did for the Fano plane, we can easily translate this into a BIBD by considering the lines as blocks, and we get that $r = n + 1$, $k = n + 1$, and $\lambda = 1$. Then, using the equations in 1.4, we get

$$r(k-1) = \lambda(v-1) \implies v = 1 + \frac{(n+1)n}{1} = n^2 + n + 1,$$

then

$$vr = bk \implies b = \frac{(n^2+n+1)(n+1)}{(n+1)} = n^2 + n + 1.$$

Therefore, if a finite projective plane of order n exists, it is equivalent to a $(n^2+n+1, n+1, 1)$-design. It turns out that such a design exists whenever n is a prime power, but constructing this design is outside the scope of this paper. It is an open problem as to whether a finite projective plane of order n exists for some n that is not a prime power.

References

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