Rational Catalan Numbers

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ABSTRACT. In this paper, we will discuss mathematical properties of Rational Catalan Numbers.

1 Introduction

Rational Catalan numbers, which we will write as $R(m, n)$ in this paper, count the number of paths through an $m \times n$ grid from the bottom left corner to the top right corner that stay below the diagonal and only involve movements up and to the right, where gcd(m,n)=1. We will also define an (m, n)-Dyck path as a valid path counted in $R(m, n)$.

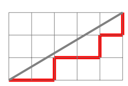

Figure 1: Example of a path counted in R(5,3)

In this research paper, we will explore two questions about Rational Catalan numbers: 1. What is a generalized formula for Rational Catalan numbers? 2. What other mathematical properties do Rational Catalan numbers count?

2 Basic Theorems of Rational Catalan Numbers

		m				
		1	2	3	4	5
	1	x	1	1	1	1
	2	1	x	2	x	3
n	3	1	2	x	5	7
	4	1	x	5	x	14
	5	1	3	7	14	x

Figure 2: Table with $R(m, n)$ for values of m and n between 1 and 5

Theorem 2.1. $R(m, n) = R(n, m)$.

Any (m, n)-Dyck path can be reflected across the line $y = x$ to form an (n, m)-Dyck path.

Theorem 2.2. $R(n, n) = R(n, n + 1)$.

Since each $(n, n + 1)$-Dyck path below the line $y = \frac{n+1}{n}x$ involves going up as the final step, the amount of $(n, n + 1)$-Dyck paths is equivalent to the amount of paths from $(0, 0)$ to (n, n) lying below the line $y = \frac{n+1}{n}x$. The set of points lying below or on the line $y = x$ is equivalent to the set of points lying below or on the line $y = \frac{n+1}{n}x$ excluding the point $(n, n + 1)$. Since the last move of a $(n, n + 1)$-Dyck path is always the up move, the amount of (m, n)-Dyck paths from $(0, 0)$ to $(n, n + 1)$ is the same as the amount of Dyck paths from $(0, 0)$ to (n, n), so $R(n, n + 1) = R(n, n)$.

3 A Generalized Formula

Definition 3.1. For relatively prime m and n, define $P(m, n)$ as the set of all permutations of m R's and n U's, which is the set of all paths from $(0, 0)$ to (m, n).

Definition 3.2. If we let T be an arbitrary permutation in $P(m, n)$, define $\mathrm{Cy}(M)$ as the set of all possible permutations P' that are **cyclic shifts** of T, including T itself.

We can visualize the elements of $\mathrm{Cy}(T)$ by first graphing T on a $2m \times 2n$ grid as a path from $(0, 0)$ to (m, n), and additionally graphing T as a path from (m, n) to $(2m, 2n)$. All the permutations in $\mathrm{Cy}(T)$ can be represented as subpaths of the total path from $(0, 0)$ to $(2m, 2n)$ of length $m + n$. We will refer to this path from $(0, 0)$ to (m, n) as the **double-path** of T.

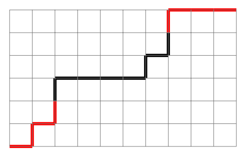

Figure 3: A cycle counted in $\mathrm{Cy}(RURUURRR)$ highlighted in black, and the rest of the double path in red.

Theorem 3.3. $R(m, n) = \frac{(m+n-1)!}{m! * n!}$

Let R be an arbitrary element in $P(m, n)$. Notice that a path counted in $\mathrm{Cy}(R)$ that starts at (a, b) in the double-path of R is a Dyck path if and only if the line $y = \frac{nx}{m} + (b - \frac{na}{m})$ does not intersect the double-path of R at any points other than (a, b) and $(a + m, b + n)$. If the line $y = \frac{nx}{m} + (b - \frac{na}{m})$ did intersect the double path at a point other than (a, b) and $(a + m, b + n)$, it would also intersect the original path at a point in between (a, b) and $(a + m, b + n)$, making the path a non-Dyck path.

Over all points in the double-path with $0 \leq a \leq m$ and $0 \leq b \leq n$, there would be a unique point (a, b) where $(b - \frac{na}{m})$ is maximized, since m and n are relatively prime (as stated in the definition of Rational Catalan numbers earlier in the paper), making this line higher than any other parallel line of the form $y = \frac{nx}{m} + (b - \frac{na}{m})$. This would cause the line $y = \frac{nx}{m} + (b - \frac{na}{m})$ to not intersect the double-path at points other than (a, b) and $(a + m, b + n)$. In addition, any other path starting at (x, y) would intersect the Dyck path from (a, b) to $(a + m, b + n)$ at some non-lattice point, meaning that all the other paths are not Dyck paths. This means that for each permutation R, $Cy(R)$ contains one valid Dyck path.

Since m and n are relatively prime, $Cy(R)$ contains $m+n$ elements for all permutations $R \in P(m,n)$. Since there are $|P(m,n)| = \binom{n+m}{m}$, and $\frac{1}{m+n}$ of those elements represent Dyck paths, the amount of $(m,n)-$Dyck paths is $\binom{m+n}{m}\frac{1}{m+n} = \frac{(m+n-1)!}{m!*n!}$.

For a better visual representation of this, all of the lines of the form $y = \frac{nx}{m} + (b - \frac{na}{m})$ are depicted in orange below.

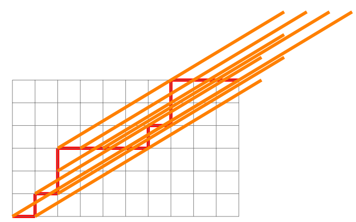

Figure 4: The parallel lines between points in the double-path depicted in orange

As you can see, the top orange line is the only orange line that does not intersect the Dyck path at more than two points.