Pattren Avoidance in Permuatations

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Definition (Pattern Avoidance in Permutations). Let p be a permutation, and let q be a permutation of length k. We say that p *contains* q as a pattern if there exist indices $i_1 < i_2 < \cdots < i_k$ such that, for all $1 \leq a, b \leq k$, $p_{i_a} < p_{i_b}$ if and only if $q_a < q_b$. If no such indices exist, we say that p *avoids* q.

Example Assume that we have a permutation $p = (4, 2, 3, 1)$ and a pattern $q = (1, 2, 3)$ and there is no subsequence of p in increasing order. Thus, p avoids q.

Example Assume that we have a permutation $p = (5, 2, 3, 4, 1)$ and a pattern $q = (2, 1, 3)$ and there is no subsequence (a, b, c) of p such that $b < a < c$. Thus, p avoids q.

Definition (Reverse of Permutations). We define reverse of a permutation as $f : S_n \to S_n$ such that $f(\pi_1, \pi_2, \cdots \pi_n) = (\pi_n, \pi_{n-1}, \cdots \pi_1)$ where $f(f(\pi)) = f(\pi_n, \pi_{n-1}, \cdots \pi_1) = (\pi_1, \pi_2, \cdots \pi_n) = \pi$. Thus, $f : S_n \to S_n$ is it's own Inverse so it's a Bijective function.

Example We have $f(1, 2, 3, \cdots n) = (n, n-1, n-2, \cdots 1)$ and $f(n, n-1, n-2, \cdots 1) = (1, 2, 3 \cdots n)$.

Definition (Complement of Permutations). We define complement of a permutation as $g : S_n \to S_n$ such that $g(\pi_1, \pi_2, \cdots \pi_n) = (n+1-\pi_1, n+1-\pi_2, \cdots n+1-\pi_n)$ where $g(g(\pi)) = g(n+1-\pi_1, n+1-\pi_2, \cdots n+1\pi_n) = (\pi_1, \pi_2, \cdots \pi_n) = \pi$. Thus, $g : S_n \to S_n$ is it's own Inverse so it's a Bijective function.

Example We have $g(1, 3, 2) = (3, 1, 2)$ and $g(3, 1, 2) = (1, 3, 2)$

1 Patterns of Length 2 and Length 3

The only possible permutations of length 2 are $(1,2)$ and $(2,1)$. If we have any permutation $\pi = (\pi_1, \pi_2, \cdots \pi_n)$ then it will contain pattern $(1,2)$ if there is a subsequence of length 2 in increasing order and contain pattern $(2,1)$ if there is a subsequence of length 2 where the elements are decreasing. The only permutation that avoids pattern $(1,2)$ is the permutation of decreasing order of n. The only permutation which avoids pattern $(2,1)$ is the increasing order permutation of n. We define $\mathbf{S_n(q)}$ to be total number permutations of $[n]$ which avoids pattern q.Thus,$S_n(1,2) = S_n(2,1) = 1$.

Lemma 1: $S_n(123) = S_n(321)$
If we have any permutation $\pi = (\pi_1, \pi_2, \cdots \pi_n)$ and contain pattern 123 then there will be a subsequence of length 3 $(\pi_i \pi_j \pi_k)$ in increasing order and if we reverse π we will have $\pi' = (\pi'_1 \pi'_2 \cdots \pi'_n)$ where $\pi' i = \pi_{n-i+1}$ so we have a subsequence of π' $(\pi'_{n-k+1} \pi'_{n-j+1} \pi'_{n-i+1}) = (\pi_k, \pi_j, \pi_i)$ which is a decreasing subsequence so π' follow pattern (321).The function of mapping a permutation to it's reverse is a Bijective function as this function is it's own inverse.Thus, total number of permutations which follow pattern (123) is equal to permutations that follow pattern (321). Therefore, $S_n(123) = S_n(321)$

Lemma 2: $S_n(132) = S_n(231) = S_n(312) = S_n(213)$

We have $f : S_n \to S_n$ which map any permutation to its reverse and $g : S_n \to S_n$ which map any permutation to its complement.If there is a permutation $\pi = (\pi_1, \pi_2, \cdots \pi_n)$ and contain pattern 132 then there is a subsequence (π_i, π_j, π_k) where $i < j < k$ and $\pi_i < \pi_k < \pi_j$ so $f(\pi) = \pi' = (\pi'_1, \pi'_2 \cdots \pi'_n)$ where $\pi'_i = \pi_{n-i+1}$ and we have a subsequence $(\pi'_{n-k+1}, \pi'_{n-j+1}, \pi'_{n-i+1}) = (\pi_k, \pi_j, \pi_i)$ which contain pattern $(2,3,1)$. Thus, if a permutation avoid pattern $(1,3,2)$ then its reverse will avoid pattern $(2,3,1)$. Therefore, $S_n(132) = S_n(231)$. $g(\pi') = \sigma = (\sigma_1, \sigma_2 \cdots \sigma_n)$ where $\sigma_i = n+1-\pi'_i$ and π' contain pattern $(2,3,1)$ so for some $i < j < k$ we have $\pi'_k < \pi'_i < \pi'_j$ so we have a subsequence $(\sigma_i, \sigma_j, \sigma_k) = (n+1-\pi'_i, n+1-\pi'_j, n+1-\pi'_k)$ and it contains pattern $(2,1,3)$. Thus, if a permutation avoids pattern $(2,3,1)$ then its complement will avoid pattern $(2,1,3)$. Therefore , $S_n(231) = S_n(213)$. We have $f(\sigma) = \sigma' = (\sigma'_1, \sigma'_2 \cdots \sigma'_n)$ where σ contains pattern $(2,1,3)$ so we have $i < j < k$ such that $\sigma_j < \sigma_i < \sigma_k$ and we have a subsequence $(\sigma'_{n-k+1}, \sigma'_{n-j+1}, \sigma'_{n-i+1}) = (\sigma_k, \sigma_j, \sigma_i)$ so it contains pattern $(3,1,2)$.Thus, if a permutation avoids pattern $(2,1,3)$ then its reverse will avoid pattern $(3,1,2)$. Therefore, $S_n(312) = S_n(213)$.We have proved that $S_n(312) = S_n(213) = S_n(231) = S_n(132)$

Lemma 3

For all positive integers n, we have $S_n(123) = S_n(132)$.

Proof. There are several ways to prove this first nontrivial result of the subject. The one we present here is due to R. Simion and F. Schmidt. Recall that an entry of a permutation which is smaller than all the entries that precede it is called a *left-to-right minimum*. Note that the left-to-right minima form a decreasing subsequence.

We are going to construct a bijection f from the set of all 132-avoiding n-permutations to the set of all 123-avoiding n-permutations which leaves all left-to-right minima fixed.

The map f is defined as follows. Keep the left-to-right minima of p fixed, and write all the other entries in decreasing order. The obtained permutation $f(p)$ is always 123-avoiding as it is the union of two decreasing subsequences: one of them is the sequence of all left-to-right minima, and the other is the decreasing sequence into which we arranged the remaining entries.

Example If $p = 67341258$, then the left-to-right minima of p are $6, 3$, and 1, therefore

$$f(p) = 68371542.$$

We would like to point out that the left-to-right minima of p and $f(p)$ are the same, even if some other entries of p have moved. Indeed, we can say that f simply rearranges the m entries that are not left-to-right minima pair by pair. That is, whenever we (impersonating the function f) see a pair of these entries that is not in decreasing order, we swap them. This algorithm stops in at most m steps. Moreover, each step of this algorithm moves a smaller entry to the right and a larger one to the left, and therefore never creates a new left-to-right minimum.

We note that this is the only 123-avoiding permutation with the given set and position of left-to-right minima. Indeed, if there were two entries x and y that are not left-to-right minima and form a 12-pattern, then the left-to-right minimum z that is closest to x on the left, and the entries x and y would form an increasing sequence.

Now we prove that f is a bijection by showing that it has an inverse. Let q be an n-permutation that avoids 123. Keep the left-to-right minima of q fixed, and fill in the remaining positions with the remaining entries, moving left-to-right, as follows. At each step, place the smallest element not yet placed which is larger than the closest left-to-right minimum on the left of the given position. Call the obtained permutation $g(q)$.

Example If $q = 68371542$, then the left-to-right minima of q are $6, 3$, and 1. To the empty slot between 6 and 3, we put the smallest of the two entries that are larger than 6, that is, 7. To the empty slot between 3 and 1, we put the smallest entry not used yet that is larger than 3, that is, 4. Immediately on the right of 1, we put the smallest entry not used yet that is larger than 1, that is, 2. We finish this way, by placing 5 and 8 to the remaining slots, to get

$$g(q) = 67341258.$$

The obtained permutation is always 132-avoiding. Indeed, if there were a 132- pattern in it, then there would be one which starts with a left-to-right minimum, but that is impossible as entries larger than any given left-to-right minimum are written in increasing order.

Note again that $g(q)$ is the only 132-avoiding permutation that has the same set and position of left-to-right minima as q. Indeed, if at any given instance, two entries $u < v$ that are larger than a left-to-right minimum a were in decreasing order, then $a\,v\,u$ would be a 132-pattern.

This proves that $g(f(p)) = p$, implying that f is a bijection, and proving our theorem.

Theorem 4

For all positive integers n, we have $S_n(132) = S_n(123)$.

Let $c_n = S_n(132)$. Suppose we have a 132-avoiding n-permutation in which the entry n appears in the ith position. Then any entry to the left of n must be larger than any entry to the right of n; otherwise, if x is on the left of n and y is on the right with $x < y$, the triple $x\,n\,y$ would form a 132-pattern.

Thus, the entries to the left of n must be

$$\{\, n-i+1,\, n-i+2,\, \ldots,\, n-1 \,\},$$

and the entries to the right must be the set $[\, n-i \,] = \{1, 2, \ldots, n-i\}$. There are c_{i-1} possibilities for ordering the entries on the left and c_{n-i} possibilities for ordering the entries on the right.

Summing over all i gives the recursion

$$c_n = \sum_{i=0}^{n-1} c_{i-1}\, c_{n-i}. \tag{4.1}$$

Let

$$C(x) = \sum_{n \geq 0} c_n x^n$$

be the ordinary generating function of the sequence (c_n). Then (4.1) implies

$$C(x) = 1 + x\, C(x)^2. \tag{4.2}$$

Solving this quadratic equation for $C(x)$ yields

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}. \tag{4.3}$$

Thus the sequence (c_n) is the sequence of Catalan numbers, which establishes the theorem.

Corollary 5

Using Lemma 1 , Lemma 2 , Lemma 3 and Theorem 4 we have proved that total number of permutations of length n which avoid any three letter pattern is equal to the nth Catalan number.

2 Refrences

1. Simion, R., & Schmidt, F. W. (1985). *Restricted permutations. European Journal of Combinatorics*, **6**(4), 383–406.

2. Bóna, M. (2004). *Combinatorics of permutations.* Chapman & Hall/CRC.