PATTERN AVOIDANCE IN PERMUTATIONS

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Abstract. This exposition introduces the combinatorial theory of pattern avoidance in permutations, a topic that unites structural and enumerative ideas across modern combinatorics. Beginning with the fundamental notions of permutation containment and avoidance, we explore how closure operations, growth rates, and Catalan-number enumerations emerge naturally within this framework. The discussion then turns to Wilf equivalence, which formalizes when distinct patterns lead to identically enumerated classes, revealing hidden symmetries in the combinatorial landscape. Finally, we outline several connections between pattern avoidance and areas of mathematical physics, including lattice models, the Temperley–Lieb algebra, and perturbative quantum field theory.

1. Introduction

Permutation patterns offer one of the simplest yet most fascinating gateways into modern combinatorics [1–4]. At first glance, permutations are merely reorderings of numbers, but behind this simplicity lies a remarkably rich structure.

By examining which smaller arrangements, or patterns, appear inside a permutation, we uncover a deep world of combinatorial symmetry, enumeration, and algebraic behavior [3].

The central idea is pattern avoidance [5–9]: a permutation is said to avoid a smaller pattern if no subsequence of its entries has the same relative order as that pattern. This deceptively simple definition turns out to connect with a wide range of mathematical and physical ideas, from Catalan number combinatorics [10, 11] and Young tableaux [12] to algebraic structures like the Temperley Lieb algebra, statisical mechanics [13] and even renormalization in quantum field theory [14–16].

The study of pattern avoidance thus sits at a crossroads between enumeration, structure, and symmetry [3]. Enumeratively, one asks: How many permutations of size n avoid a given pattern?

Structurally, one investigates how such permutations can be decomposed or characterized by operations such as sums and skew sums.

And symmetrically, one seeks to understand when different forbidden patterns lead to the same counting sequence, a phenomenon known as Wilf equivalence [17].

In what follows, we explore these ideas through several concise sections. Section 1 introduces the basic definitions of permutation containment and avoidance, together with closure

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properties and the notion of growth rate. Section 2 focuses on the simplest nontrivial examples; patterns of length three, showing how their enumeration leads naturally to the Catalan numbers. Section 3 develops the idea of Wilf equivalence, formalizing when two avoidance classes are combinatorially identical. Finally, the Appendix outlines a few surprising applications in physics, illustrating how similar exclusion principles and diagrammatic rules appear in statistical mechanics and quantum field theory.

2. Pattern avoidance in permutations

The study of pattern avoidance [1] begins with the basic notion of a permutation, which provides the combinatorial playground [2] for everything that follows.

Definition 2.1. A permutations is a linear ordering of the elements of the set $[n] = \{1, 2, 3, 4...n\}$. If it consists of n entries then, it is also called an n permutation.

Every permutation can be seen as a rearrangement of the numbers 1 through n, and thus as a combinatorial object 1 through n, and thus as a combinatorial object encoding relative order.

To understand more subtle relationships between permutations, we introduce the idea of containment.

Definition 2.2. A permutation π is said to contain a permutation σ of length k that is order isomorphic to σ, i.e, that it has the same pairwise comparisons as σ.

Permutation containment is a partial order on the set of all finite permutations, so if σ is contained in π, we write $\sigma \leq \pi$. If $\sigma \not\leq \pi$ then, we say π avoids σ.

This notion of avoidance allows us to speak of permutation classes [18]- families of permutations that are closed under taking sub patterns.

Remark 2.3. If C is a class containing the permutation π and $\sigma \leq \pi$then σ must also lie in C.

Hence, given any set X of permutations, one can obtain a class simply by taking all smaller permutations contained in the elements of X.

$$\mathrm{Sub(X)} = \{\text{œ} : \text{œ} \leq \text{ß}, \text{ß} \in \mathrm{X}\}. \tag{2.1}$$

A more familiar and fruitful way to define a class is by avoidance:

$$\mathrm{Av(B)} = \{\text{ß} : \text{ßavoidsallfi} \in \mathrm{B}\}. \tag{2.2}$$

Such a class is called the avoidance class of B.

Now, in practice it is often quite difficult to compute generating functions of permutation classes, and thus we must content ourselves with the rough asymptotics of $|C_n|$. To do so, we define the *upper growth rate* and the *lower growth rate* of the class C by

$$\overline{gr}(C) = \lim_{n\to\infty} \mathrm{sup(C_n)^{1/n}}; \underline{\mathrm{gr}}(\mathrm{C}) = \mathrm{lim_{n\to\infty}inf(C_n)^{1/n}} \tag{2.3}$$

Conjecture 2.4. *For every permutation class C, the upper and lower growth rates coincide.*

When they do, their common value is called the *growth rate* of the class.

A convenient way to build larger permutations from smaller ones is through sum and skew sum operations. These constructions allow us to describe the internal structure of a permutation class and understand its closure properties.

Definition 2.5. If π has a length k and σ has length l, we can also define the sum of π and σ by

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & i \in [1, k] \\ \sigma(i-k)+k & i \in [k+1, k+l] \end{cases} . \tag{2.4}$$

We can analogously define a skew sum as

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i)+l & \in [1, k] \\ \sigma(i-k) & i \in [k+1, k+l] \end{cases} \tag{2.5}$$

These operations combine permutations in “ascending” or “descending” fashion, respectively.

Some classes remain stable under these operations, motivating the following definition.

Definition 2.6. The permutation class C is said to be sum closed (respectively, skew closed) if $\pi \oplus \sigma \in C$ (respectively, $\pi \ominus \sigma \in C$) for every pair of permutations $\sigma, \pi \in C$. The permutation π is further said to be *sum decomposable* if it can be expressed as a nontrivial sum (respectively, skew sum) of permutations , and sum (respectively, skew) indecomposable otherwise. It is easy to establish that a class is sum (respectively, skew) closed if and only if all of its basis elements are sum (respectively, skew) indecomposable.

Because no permutation can be both sum- and skew-decomposable, we obtain the following observation.

Remark 2.7. Every principal permutation class is either sum or skew closed.

To understand the enumerative consequences of such closure properties, we recall a classical lemma due to Fekete.

Definition 2.8. The sequence $\{a_n\}$ is said to be super-multiplicative if $a_{m+n} > a_m a_n$ $\forall m$ and n.

Lemma 2.9. ***Fekete's lemma:-*** *If the sequence* $\{a_n\}$ *is super-multiplicative then* $\lim(\mathrm{a_n})^{1/\mathrm{n}}$ *exists and is equal to* $\sup(\mathrm{a_n})^{1/\mathrm{n}}$.

This lemma, though simple, plays a powerful role in proving the existence of growth rates for closed classes.

Proposition 2.10. *Every sum closed (or, by symmetry, skew closed) permutation class has a (possibly infinite) growth rate. In particular, this holds for every principal class.*

Proof. Suppose C is a sum closed permutation class, and thus $\pi \oplus \sigma \in C_{m+n}$ for all $\pi \in C_{m+n}$ and $\sigma \in C_{m+n}$. Moreover, a given $\tau \in C_{m+n}$ arises in this way from at most one such pair so $|C_{m+n}| \geq |C_m||C_n|$. This shows that the sequence $\{|C_n|\}$ is supermultipicative, and thus $\lim(\mathrm{C_n})^{1/\mathrm{n}}$ exists by Fekete's Lemma. ■

Finally, sum decomposition provides a natural way to break down a permutation into simpler building blocks.

Definition 2.11. For every permutation π there are unique sum indecomposable permutations $\alpha_1, ..., \alpha_k$ (called the sum components of π) such that $\pi = \alpha_1 \oplus ... \oplus \alpha_k$.

Definition 2.12. A permutation is *layered* if it is the sum of decreasing permutations

Layered permutations play an important role in understanding pattern avoidance, as they naturally arise in classes defined by small forbidden patterns.

They also serve as the bridge to the next section, where we study the simplest and most fundamental avoidance problem, avoiding a pattern of length three.

3. Avoiding a Permutation of length three

The simplest non-trivial examples of pattern avoidance occur when the forbidden pattern has length three.

There are six possible permutations of three elements, but many of them behave in similar ways under symmetry (reverse, complement, inverse).

Thus, studying just a few representative cases already reveals the key combinatorial ideas at play.

Among these, the patterns 231 and 321 hold special significance.

Both lead to beautiful enumerative results connected with the Catalan numbers, one of the most pervasive sequences in combinatorics.

Theorem 3.1. *The class* Av(231) *is counted by Catalan numbers.*

Proof. Let $\pi \in \mathrm{Av_n}$ and let the maximum element n appear at position k. We write π as

$$\pi = \alpha n \beta, \tag{3.1}$$

where α consists of numbers less than n and appearing before n; β consists of numbers less than n and appearing after n.

We claim that all numbers in α are less than all numbers in β. This is because if there existed a $a \in \alpha$ and $b \in \beta$ with $a > b$ then the pattern anb would be a 231 pattern. Thus,$\alpha \in \mathrm{Av_i}(231)$, $\beta \in Av_{n-1}(231)$ for some $i = |\alpha|$. Therefore,

$$|\mathrm{Av_n}(231)| = \sum_{\mathrm{i=0}}^{\mathrm{n-1}} |\mathrm{Av_i}(231)| \cdot |\mathrm{Av_{n-1-i}}(231)|. \tag{3.2}$$

This is exactly the **Catalan recurrence:-**

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}. \tag{3.3}$$

With $|\mathrm{Av_0}(231)| = 1$, we get:

$$|\mathrm{Av_n}(231)| = \mathrm{C_n}. \tag{3.4}$$

∎

The next natural question is whether other length-three patterns lead to similar enumerations. Surprisingly, several of them do; a phenomenon that later motivates the notion of Wilf equivalence.

Before introducing that concept, we illustrate it through another example.

A powerful combinatorial tool that connects permutations to tableaux is the Robinson–Schensted–Knuth correspondence (RSK).

Theorem 3.2. *The length of the longest decreasing subsequence in a permutation equals the number of rows in its insertion tabelau.*

We now have

Theorem 3.3. *The class* Av(321) *is counted by Catalan numbers.*

Proof. Now, $\pi \in \mathrm{Av}(321)$ implies that the longest decreasing subsequence length of π is $\leq$ 2. Thus, π corresponds to a standard Young tabelue with atmost 2 rows, containing n boxes.

A Young diagram with atmost two rows and the total size n is uniquely determined by the size of the top row, which may be k, giving shape:

$$(k, n-k). \tag{3.5}$$

And, the number of standard Young tabuelux of shape $(k, n-k)$ is $\frac{n!}{k!(n-k+1)!}$. Summing over all valid k, we get $|\mathrm{Av_n}(321)| = \mathrm{C_n}$. ∎

The fact that both Av(231) and Av(321) are enumerated by the same Catalan numbers hints at a deeper combinatorial symmetry between the two patterns.

This observation leads naturally to the idea of Wilf equivalence, explored in the next section, which formalizes when two pattern classes are "counting the same things in disguise".

4. Wilf Equivalence

The previous section revealed a striking coincidence: both the classes Av(231) and Av(321) are counted by Catlan numbers.

At first glance, these two patterns look quite different, yet their avoidance classes have identical enumeration.

This prompts a natural question: when do two permutation patterns give rise to the same counting sequence? The study of this phenomenon leads us to the concept of Wilf equivalence.

Two permutation classes may appear unrelated, but if they contain the same number of permutations of each length, they are enumeratively indistinguishable.

Such classes are considered equivalent in the sense of Wilf.

Definition 4.1. The classes C and D are said to be *Wilf equivalent* if they are equinumerous , i.e if $|C_n|= |D_n|$ for every n.

Wilf equivalence captures a kind of hidden symmetry among permutation patterns.

For instance, reversing or complementing a pattern often preserves the size of its avoidance class, creating families of trivially Wilf-equivalent patterns.

However, there are also nontrivial equivalences that reflect deeper structural correspondences, which we shall begin to explore.

A refined version of this idea arises when we consider Ferrers boards and rook placements, which translate permutation avoidance into a geometric form.

This allows for a more nuanced comparison between patterns.

Definition 4.2. *Full rook placements (frps)* consist of Ferres boards with a designated set of cells, called rooks, so that each row and column contains precisely one rook.

Each FRP encodes a permutation: the position of the rook in row i marks the value of $\pi(i)$.

Avoiding a given permutation pattern now becomes the condition that no subset of rooks forms that pattern within the board.

Definition 4.3. We say that permutations β and γ are *shape Wilf equivalent* if given any shape λ, the number of β-avoiding frps of shape λ is the same as the number of γ-avoiding frps of shape λ.

Shape Wilf equivalence is therefore a stronger condition than ordinary Wilf equivalence: it requires the enumerative equality to hold within each geometric shape, not merely in total.

Shape Wilf equivalence behaves nicely under certain operations on permutations, particularly the sum and skew-sum constructions introduced earlier.

This leads to the following useful proposition.

Proposition 4.4. *If β and γ are shape Wilf equivalent, then for every permutation, δ, $\beta \oplus \delta$ and $\gamma \oplus \delta$ are also shape Wilf equivalent.*

Proof. Suppose that there is a bijection between β avoiding and γ avoiding frps of every shape. Now, fix a shape λ. We construct a bijection between $\beta \oplus \delta$ avoiding frps of shape λ. We construct a bijection between $\beta \oplus \delta$ avoiding frps and $\gamma \oplus \delta$ avoiding frps of shape λ. Let R be a $\beta \oplus \delta$ avoiding frps of shape λ.

We call a cell of R *dangerous* if there is a copy δ completely contained in the region above and to the right of the cell. The entire set of dangerous cells is called the *danger zone*. The *danger zone* forms a (possibly empty) Ferres board nesteled in the bottom-left corner of R. Ignoring the rookless rows and columns of the danger zone, we thus obtain a β-avoiding frp. We may then use the bijection between β-avoiding and γ-avoiding frps of that shape to produce a $\gamma \oplus \delta$ avoiding frp of shape λ as desired. ■

This property shows that Wilf-type equivalences behave functorially: they persist when larger structures are built from smaller ones.

The simplest examples arise from monotone patterns.

Theorem 4.5. *For every value of k, the permutations $k...21$ and $12...k$ are shape Wilf-equivalent.*

In words, the fully decreasing and fully increasing patterns of the same length are equivalent from this perspective — a reflection of the symmetry between order and reverse order in the permutation lattice.

A more specific instance connects back to our earlier results.

Theorem 4.6. *The permutations 321 and 231 are shape Wilf-equivalent.*

Thus, our earlier observation that both Av(321) and Av(231) are counted by the Catalan numbers is not accidental- it follows from a deeper geometric equivalence between the patterns themselves.

Not all equivalences, however, extend to the shape level.

The following example shows that ordinary Wilf equivalence can hold even when the shape version fails.

Theorem 4.7. *The permutations 1342 and 2413 are Wilf-equivalent, but not shape Wilf equivalent.*

This distinction illustrates that pattern avoidance has multiple layers of symmetry, some purely enumerative, others structural and geometric. Understanding these relationships continues to be an active and elegant area of modern combinatorics.

The combinatorial structures developed so far; pattern classes, growth rates, and Wilf equivalences; are not merely abstract curiosities. They often emerge naturally in physical and algebraic contexts, where exclusion rules or diagrammatic constraints mirror pattern-avoidance conditions.

The appendix that follows outlines a few of these surprising connections, from statistical mechanics to quantum field theory.

5. Conclusions

The study of pattern avoidance in permutations provides a remarkable meeting point between structure, symmetry, and enumeration.Starting from the simple act of forbidding a small pattern, we encounter a wealth of mathematical phenomena: recursive decompositions, Catalan-number enumerations, and deep equivalences such as Wilf symmetry.

These ideas demonstrate how intricate combinatorial behavior can emerge from elementary definitions. Beyond pure combinatorics, the language of pattern avoidance finds echoes across mathematics and physics. Constraints that forbid certain local configurations—whether in lattice models, algebraic structures, or Feynman diagrams—mirror the same exclusion principles that govern permutation classes.

Thus, what begins as a question about order patterns naturally extends to questions about structure, representation, and constraint in far broader settings. In essence, pattern avoidance serves as both a combinatorial laboratory and a unifying metaphor: a small, precise framework that continues to illuminate connections between counting, geometry, and the physical world.

Appendix: Applications of Pattern Avoidance in Physics

This appendix outlines several ways in which the study of pattern avoidance in permutations appears naturally in models of statistical mechanics, two-dimensional lattice systems, and perturbative quantum field theory. The aim is expository rather than exhaustive, emphasizing the conceptual links rather than technical formulations.

1. Pattern Avoidance in Statistical Mechanics. Many systems in statistical mechanics impose local exclusion constraints. For example, in a hard-core lattice gas model, particles are not allowed to occupy adjacent sites. More generally, physical configurations are often restricted by rules that forbid certain local geometric patterns.

This can be encoded combinatorially by pattern avoidance in permutations: a permutation encodes the ordering or arrangement of sites or interactions, while avoidance of a specific pattern records that some local configuration is prohibited. Counting pattern-avoiding permutations thus corresponds to counting the number of physically admissible states in the constrained statistical system. In this way, classical enumerative results (such as Catalan numbers for Av(321)) describe growth rates of allowed state spaces in constrained lattice and gas models.

2. Temperley–Lieb Algebra and 321-Avoiding Permutations. The Temperley Lieb algebra arises in the study of a variety of exactly solvable two-dimensional statistical models, including the Potts and Ising models and loop models on a planar lattice. A standard basis of this algebra consists of non-crossing matchings drawn between n boundary points.

A key combinatorial fact is that permutations avoiding the pattern 321 are in bijection with these non-crossing matchings. Thus, the class Av(321) provides a purely combinatorial model for the same algebraic and diagrammatic structures used to solve two-dimensional lattice models. The connection offers a correspondence between planarity constraints in statistical mechanics and pattern avoidance in permutations.

3. Pattern Avoidance in Quantum Field Theory. In perturbative quantum field theory, observables are computed using expansions over Feynman diagrams. Not all diagrams contribute: certain diagrams are excluded during renormalization because they contain subdivergences or forbidden substructures.

This elimination can be described using pattern avoidance. The diagrams may be indexed in a canonical way that translates the presence of a forbidden subgraph into the presence of a forbidden permutation pattern. In this light, renormalization can be seen as a process of removing pattern-containing structures from a combinatorially defined sum over diagrams. This point of view is closely related to the Hopf algebra of Feynman diagrams introduced by Connes and Kreimer and provides a conceptual bridge between combinatorics and renormalization theory.

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