AUTOMATIC SEQUENCES AND TRANSCENDENTAL NUMBERS

NEIL KOLEKAR

1. Introduction

Automatic sequences are sequences with integer elements between 0 and $k-1$, inclusive, that can be represented by a k-DFAO (Determinstic Finite Automata with Output). While the theory of such sequences is itself broad, it has numerous applications to transcendental number theory; the target of this paper is to prove many theorems that develop a strong connection between these two areas and and then demonstrate the use of Automata Theory to proving the algebraicity and transcendence of formal power series over finite fields. This is mainly because automatic sequences are central to characterizing formal power series that are algebraic over a finite field $\mathbb{F}_p(X)$.

Automatic sequences were introduced by Julius Büchi in 1960, although he introduced it in a more logical and theoretical way that did not use terminology from Automata Theory. In 1972, Cobham further studied these sequences and called these sequences uniform tag sequences.

Definition 1.1. A k-DFAO (Deterministic Finite Automata with Output) is a 6-tuple $(Q, \Sigma_k, \delta, q_0, \Delta, \tau)$, where

- Q is a finite set called the set of *states*,
- Σ_k is the set of strings whose elements are $\{0, 1, 2, \dots, k-1\}$ and are in base k,
- $\delta : Q \times \Sigma_k \to Q$ is called the *transition function*,
- $q_0 \in Q$ is called the *starting state*,
- Δ is a finite set called the *output alphabet*,
- $\tau : Q \to \Delta$ is the output function mapping from the set of starting states to the output alphabet.

Using the concept of a DFAO, we can generate a bounded sequence of nonnegative integers for each DFAO. The sequences are automatic sequences.

Definition 1.2. A sequence $(a_n)_{n=0}^{\infty}$ is considered *k-automatic* if and only if $a_n = \tau(\delta(q_0, [n]_k))$ for some k-DFAO $(Q, \Sigma_k, \delta, q_0, \Delta)$, where $[n]_k$ is the base-k representation of n.

Some well-known examples of automatic sequences are

- The **Thue–Morse** sequence (a_n), where a_n is the number of 1's in the binary representation of n, modulo 2.
- The **Rudin–Shapiro** sequence (r_n), where r_n is the number of 11 consecutive subsequences in the binary representation of n, modulo 2.
- The **Baum–Sweet** sequence (b_n), where $b_n = 1$ if the binary representation of n contains no block of consecutive 0's of odd length, and 0 otherwise.

Definition 1.3. We say that a map $\phi : \Sigma^\star \to \Delta^\star$ is a *morphism* if $\phi(x)\phi(y) = \phi(xy)$ for all $x, y \in \Sigma^\star$. Furthermore, we say that a morphism ϕ is *k-uniform* if for all $a \in \Sigma^\star$, $|\phi(a)| = k$.

Definition 1.4. Let $\mathbf{a} = (a_n)_{n=0}^{\infty}$ be an infinite sequence. The k-kernel of $\mathbf{a}$ is the set of subsequences defined as

$$K_k(\mathbf{a}) = \{(a_{n(k^i)+j})_{n\geq 0} : i \geq 0, 0 \geq j < k^i\}.$$

Theorem 1.5 (Projections). *If a sequence is k-automatic and $d \geq 2$ is a divisor of k, then the sequence is d-automatic as well.*

Theorem 1.6 (Cobham). *Let $k \geq 2$ and $l \geq 2$ be multiplicatively independent integers. If a sequence of integers is both k-automatic and l-automatic, then the sequence is ultimately periodic.*

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Theorem 1.7. *For any integer $k \geq 2$, the sequence $\boldsymbol{a} = (a_n)_{n=0}^{\infty}$ is k-automatic if and only if $K_k(\boldsymbol{a})$ is finite.*

2. Formal Power Series and Algebraicity over Fields

2.1. **Introduction and First Examples.** We begin this section by recalling terminology that are principal to this chapter.

Definition 2.1 (Laurent series)**.** A *formal Laurent series* of a sequence $(a_n)_{n\geq -n_0}$ for some integer n_0 is a formal power series of the form

$$f(X) = \sum_{n\geq -n_0} a_n X^n.$$

Definition 2.2. The ring $K[[X]]$ for some ground field K is the set given by

$$\left\{ \sum_{n=0}^{\infty} a_n X^n : a_n \in K \right\}.$$

The field $K((X))$ is similarly given by

$$\left\{ \sum_{n=-n_0}^{\infty} a_n X^n : n_0 \in \mathbb{Z}, a_n \in K \right\}.$$

Definition 2.3. We say that a Laurent series $F(X) = (a_n)_{n\geq -n_0}$ is algebraic over the field of rational functions $K(X)$ if for some integer d there exists polynomials $A_0(X), \ldots, A_d(X)$ with coefficients in K and not all 0 such that

$$\sum_{i=0}^{d} A_i(X) F(X)^i = 0.$$

Example. Let $T(X) = \sum_{n\geq 0} t_n X^n$ be the formal power series for the Thue–Morse sequence $(t_n)_{n\geq 0}$. Thus we may use the recurrence for the Thue–Morse sequence gto obtain

$$\begin{aligned} T(X) &= \sum_{n\geq 0} t_{2n} X^{2n} + \sum_{n\geq 0} t_{2n+1} X^{2n+1} \\ &= \sum_{n\geq 0} t_n X^{2n} + X \sum_{n\geq 0} (t_n + 1) X^{2n} \\ &= T(X^2) + XT(X^2) + \frac{X}{1 - X^2} \end{aligned}$$

and in $\mathbb{F}_2(X)$ this is equivalent to

$$(1 + X)^3 T(X)^2 + (1 + X)^2 T(X) + X = 0.$$

Therefore, $T(X)$ is algebraic over $\mathbb{F}_2(X)$.

2.2. **Christol's Theorem.** In light of the discussion in 2.1, we may ask ourselves the following question.

Question 2.4. *How can we characterize formal power series that are algebraic over a finite field $\mathbb{F}_p(X)$ for some prime p?*

In 1979, Christol was able to answer this question using automatic sequences.

Theorem 2.5 (Christol, 1979)**.** *A formal power series $\sum_{n=0}^{\infty} a_n X^n \in \mathbb{F}_p(X)$ is algebraic over $\mathbb{F}_p(X)$ if and only if $(a_n)_{n\geq 0}$ is p-automatic.*

Proof (⇐). Suppose that $(a_n)_{n\geq 0}$ is p-automatic. For each state $s \in Q$ define

$$f_s(X) := \sum_{n\geq 0, \tau(q_0,[n]_p)=s} X^n \in \mathbb{F}_p(X).$$

Suppose that the state t_i transitions to s under the letter b_i for all $1 \leq i \leq k$; it follows that

$$f_s(X) = \sum_{i=1}^{k} X^{b_i} f_{t_i}(X^p) = \sum_{i=1}^{k} X^{b_i} f_{t_i}(X)^p.$$

Let $Q = \{s_1, s_2, \ldots, s_n\}$. At this point, an induction argument is sufficient to show that

$$f_s, f_s^p, f_s^{p^2}, \ldots, f_s^{p^d} \in \langle f_{s_1}^{q^{d+1}}, f_{s_2}^{q^{d+1}}, \ldots f_{s_n}^{q^{d+1}} \rangle$$

for all $d \geq 1$, which immediately implies that $f_s(X)$ is algebraic over $\mathbb{F}_p(X)$ for all $s \in Q$. Thus,

$$f(X) = \sum_{n\geq 0} a_n X^n = \sum_{n\geq 0} \tau(\delta(q_0, [n]_p))X^n = \sum_{s\in Q} \tau(s) f_s(X),$$

so $f(X)$ is algebraic over $\mathbb{F}_p(X)$. ■

Proof (⇒). For all $1 \leq k \leq p$, let Λ_k be the $\mathbb{F}_p$-linear operator such that

$$f(X) = \sum_{n\geq 0} a_n X^n \Rightarrow \Lambda_k f(X) = \sum_{n\geq 0} a_{k+pn} X^n.$$

Lemma 1. $F(X) = \sum_{i=0}^{p-1} X^k \Lambda_k F(X^i)$.
Proof of Lemma 1. Let $F(X) = \sum_{i\geq 0} a_i X^i$. We can write

$$\begin{aligned}
F(X) &= \sum_{i\geq 0} a_i X^i \\
&= \sum_{0\leq k\leq p} \sum_{i\geq 0} a_{pi+k} X^{pi+k} \\
&= \sum_{0\leq k\leq p} X^k \sum_{i\geq 0} a_{pi+k} X^{pi} \\
&= \sum_{0\leq k<p} X^k \left(\sum_{i\geq 0} a_{pi+k} X^{pi} \right)^p \\
&= \sum_{0\leq k<p} X^k \Lambda_k (F(X))^p,
\end{aligned}$$

and we are done. □
Lemma 2. $\Lambda_k(G^p H) = G\Lambda_k(H)$.
Proof of Lemma 2. Let $G(X) = \sum_{i\geq 0} g_i X^i$ and $H(X) = \sum_{i\geq 0} h_i X^i$. Notice that

$$\Lambda_k(G^p H) = \Lambda_k \left(\left(\sum_{l\geq 0} g_l X^l \right)^p \left(\sum_{j\geq 0} h_j X^j \right) \right) = \Lambda_k \left(\left(\sum_{l\geq 0} g_l X^{pl} \right) \left(\sum_{j\geq 0} h_j X^j \right) \right).$$

Thus,

$$
\begin{aligned}
\Lambda_k(G^pH) &= \Lambda_k\Bigg(\Bigg(\sum_{i\geq 0} X^i\Bigg(\sum_{l,j\geq 0, pl+j=i} g_l h_j\Bigg)\Bigg)\\
&= \Bigg(\Bigg(\sum_{i\geq 0} X^i\Bigg(\sum_{l,j\geq 0, pl+j=pi+k} g_k h_j\Bigg)\Bigg)\\
&= \sum_{i\geq 0} X^i\Bigg(\sum_{0\leq l\leq i} g_l h_{p(i-l)+k}\Bigg)\\
&= \sum_{l\geq 0} g_l X^i\Bigg(\sum_{i\geq l} h_{p(i-l)+k} X^{i-l}\Bigg)\\
&= \sum_{l\geq 0} g_l X^i\Bigg(\sum_{i\geq 0} h_{pi+k} X^{i}\Bigg)\\
&= \Bigg(\sum_{l\geq 0} g_l X^i\Bigg)\Bigg(\sum_{i\geq 0} h_{pi+k} X^{i}\Bigg)\\
&= G\Lambda_k(H). \square
\end{aligned}
$$

If $w = \sum_{i=0}^{k} b_i q^i$, let

$$\Lambda_w = \Lambda_{b_k} \circ \Lambda_{b_{k-1}} \circ \cdots \circ \Lambda_{b_0}$$

so that

$$\Lambda_w f(X) = \sum_{n\geq 0} a_{w+p^{k+1}n} X^n.$$

(This can be shown using an induction argument on k.)

Thus, $\Lambda_w f(0) = a_w$.

Let M be a p-DFAO such that

- M has initial state $f(X)$;
- The output function τ is defined so that $\tau(g(X)) = g(0)$ for all states $g(X)$;
- The transition function δ is defined so that $\delta(g(X), [k]_p) = \Lambda_k g(X)$ for all states $g(X)$ and for all $0 \leq k < p$.

Since we have previously shown that M produces the sequence $(a_n)_{n\geq 0}$, it remains to choose such an M with a finite number of states.

Suppose that $f(X) \in \mathbb{F}_p(X)$ is algebraic over $\mathbb{F}_p(X)$. Then, there exist formal power series $A_1, \ldots, A_d$ for some integer $d \geq 2$ with coefficients in $\mathbb{F}_p$ not all 0 such that

$$\sum_{k=1}^{d} A_i(X) f^{p^k} = 0,$$

since $f, f^p, f^{p^2}, \ldots$ cannot be all linearly independent over $\mathbb{F}_p(X)$.

Let $B = \max_i \deg A_i(X)$ and C be the $\mathbb{F}_p$-vector space spanned by $h_i(X) f(X)^{p^i}$ where $h_i(X) \in \mathbb{F}_p(X)$ of degree at most B for all $0 \leq i \leq d$. We can write

$$\Lambda_k\Bigg(\sum_{i=1}^{d} h_i f^{p^i}\Bigg) = \Lambda_k\Bigg(\sum_{i=1}^{d} (h_0 A_i + h_i) f^{p^i}\Bigg) = \sum_{i=1}^{d} \Lambda_k (h_0 A_i + h_i) f^{p^{i-1}} \in C,$$

as $\deg\Lambda_k(h_0 c_i + h_i) \leq \frac{2B}{p} \leq B$. Thus, $\Lambda_k(C) \subseteq C$ for all $0 \leq k < p$.

Hence, C is has a finite dimension and therefore is a finite set. Since C is closed under δ, C is a valid set of states for M. Therefore, by letting C be the set of states, we have constructed M so that it produces the sequence $(a_n)_{n\geq 0}$, so $(a_n)_{n\geq 0}$ is p-automatic, as desired. ■

The following is a generalization of Theorem 2.5, where we consider algebraicity over $\mathbb{F}_q(X)$ and q is a prime power.

Theorem 2.6. *Let $q = p^k$ for some $k \geq 1$ and prime p. A formal power series $\sum_{n=0}^{\infty} a_n X^n \in \mathbb{F}_p(X)$ is algebraic over $\mathbb{F}_q(X)$ if and only if $(a_n)_{n\geq 1}$ is q-automatic.*

Proof. See Theorem 12.2.5 in [AS03]. ■

2.3. **Applications of Christol's Theorem.** We begin this section with a basic consequence of Christol's Theorem on one of the closure properties of algebraicity over $\mathbb{F}_p(X)$.

Theorem 2.7 (Furstenberg, 1967)**.** *If F and G are algebraic over $\mathbb{F}_p(X)$, then so is their Hadamard product $F \odot G$.*

Proof. Suppose that $F(X) = \sum_{n\geq 0} f_n X^n$ and $G(X) = \sum_{n\geq 0} g_n X^n$. By Christol's Theorem, $(f_n)_{n\geq 0}$ and $(g_n)_{n\geq 0}$ are both p-automatic, and thus $(f_n g_n)_{n\geq 0}$ is also p-automatic. Applying Christol on $(f_n g_n)_{n\geq 0}$, we obtain that $F \odot G$ is also algebraic over $\mathbb{F}_p(X)$. ■

As we saw in 2.2, Christol's Theorem is the statement that allows us to link automatic sequences to the algebraicity of formal power series. Thus, we may use automatic sequences to derive results on algebraicity and transcendence over $\mathbb{Q}(X)$. In order to do so we first prove Theorem 2.9 connecting algebraicity over $\mathbb{Q}(X)$ to $\mathbb{F}_p(X)$.

Lemma 2.8. *Let $F(X)$ be a formal power series with integer coefficients, and let $F_p(X)$ be its reduction modulo p. If $F(X)$ is algebraic over $\mathbb{Q}(X)$, then $F_p(X)$ is algebraic over $\mathbb{F}_p(X)$.*

Proof. Assume to the contrary that F is algebraic over $\mathbb{Q}(X)$. There exists polynomials $A_0(X), \ldots, A_d(X)$ with coefficients in K and not all 0 such that

$$\sum_{i=0}^{d} A_i(X) F(X)^i = 0. \tag{2.1}$$

Suppose that we clear the denominators of (2.1) to obtain

$$\sum_{i=0}^{d} B_i(X) F_p(X)^i = 0 \tag{2.2}$$

where $B_i(X) \in \mathbb{Z}(X)$ for $0 \leq i \leq d$. Without loss of generality, assume that the greatest common divisor of the coefficients of all the $B_i(X)$ is 1. When reduced modulo p, not all $B_i(X)$ are reduced to 0, which implies that F_p is algebraic over $\mathbb{F}_p(X)$. ■

This leads us to the following stronger result.

Theorem 2.9. *If a formal power series $F(X)$ with integer coefficients is algebraic over $\mathbb{Q}(X)$, then the coefficients of $F(X)$ modulo p form a p-automatic sequence.*

Proof. Suppose that $F(X)$ is a formal power series with integer coefficients that is algebraic over $\mathbb{Q}(X)$. By Lemma 2.8, $F_p(X)$ is algebraic over $\mathbb{F}_p(X)$. By Christol's Theorem, the coefficients of $F(X)$ modulo p form a p-automatic sequence. ■

Example. The classical theta series $\theta_3(X) = \sum_{n\in\mathbb{Z}} X^{n^2}$ is not algebraic over $\mathbb{Q}(X)$. Assume to the contrary that it is algebraic over $\mathbb{Q}(X)$. By Theorem 2.9, the coefficients of $\theta_3(X)$ modulo 3 forms a 3-automatic sequence. However, it can be shown that

$$a_n = \begin{cases} 1 & n = 0 \\ 2 & n \text{ is a positive perfect square} \\ 0 & \text{otherwise.} \end{cases}$$

is not 3-automatic, which thereby contradicts the algebraicity of $\theta_3(X)$ over $\mathbb{Q}(X)$. $\square$

The Carlitz π is defined by

$$\pi_q(t) = \prod_{k=1}^{\infty} \left(1 - \frac{t^{q^k} - t}{t^{q^{k+1}} - t}\right)$$

for primes q. In 1941, Wade proved that π_q is transcendental over $\mathbb{F}_q(t)$. In 1990, Allouche devised a new proof using the more recent discovery of Christol's Theorem. We present the approach using Christol's Theorem.

Theorem 2.10 (Wade, 1941 and Allouche, 1990)**.** *The Carlitz π $\pi_p(t)$ is transcendental over $\mathbb{F}_p(t)$ for any prime p.*

Proof. First, we take the natural logarithm of both sides of the definition of π_p.

$$\log(\pi_p(X)) = \sum_{k=1}^{\infty} \log\left(1 - \frac{X^{p^k} - X}{X^{p^{k+1}} - X}\right).$$

Differentiating both sides of the equation yields

$$\frac{\pi_p'(X)}{\pi_p(X)} = \left(\sum_{k\geq 1} \frac{1}{X^{p^k} - X}\right) - \frac{1}{X^p - X}.$$

Define the "bracket series"[1] as below:

$$\mathcal{B} = \sum_{k\geq 1} \frac{1}{X^{p^k} - X}.$$

Assume to the contrary that π_p is algebraic over $\mathbb{F}_p(X)$. Clearly π_p' is algebraic over $\mathbb{F}_p(X)$ too, and so $\frac{\pi_p'(X)}{\pi_p(X)}$ is algebraic over $\mathbb{F}_p(X)$ as well. Since $\frac{1}{X^p - X}$ is algebraic over $\mathbb{F}_p(X)$, $\mathcal{B}$ is also algebraic over $\mathbb{F}_p(X)$. So, it suffices to show that $\mathcal{B}$ is transcendental over $\mathbb{F}_p(X)$.

We may perform the following series of algebraic manipulations:

$$\begin{aligned}
\mathcal{B} &= \frac{1}{X} \sum_{k\geq 1} \frac{1}{X^{p^k - 1}} \sum_{n\geq 0} \left(\frac{1}{X}\right)^{n(p^k-1)} \\
&= \frac{1}{X} \sum_{k\geq 1, n\geq 0} \left(\frac{1}{X}\right)^{(n+1)(p^k-1)} \\
&= \frac{1}{X} \sum_{k\geq 1, n\geq 1} \left(\frac{1}{X}\right)^{n(p^k-1)} \\
&= \frac{1}{X} \sum_{m\geq 1} \left(\frac{1}{X}\right)^{m} \sum_{k\geq 1, p^k - 1 | m} 1.
\end{aligned}$$

Thus, if we define $a_m := \sum_{k\geq 1, p^k-1|m} 1$ for all integers $m \geq 1$, then we have

$$\mathcal{B} = \frac{1}{X} \sum_{m\geq 1} \left(\frac{1}{X}\right)^m a_m.$$

By Christol's Theorem, it suffices to show that $(a_n)_{n\geq 1}$ is not automatic. Note that

$$a_{p^n - 1} = \sum_{k\geq 1, p^k - 1 | p^n - 1} 1 = d(n),$$

[1]The notational convention and naming for $\mathcal{B}$ was introduced by Wade.

where $d(n)$ is the number of divisors function. At this point, it is sufficient to show that $d(n)$ is not ultimately periodic modulo p. If $d(n)$ is periodic modulo p, then there exist integers $n_0 \geq 0$ and $t \geq 1$ such that for all $n \geq n_0$ and $i \geq 1$

$$d(n + it) \equiv d(n) \pmod{p}.$$

If we choose $i = ni'$ for some $i' \geq 1$, then

$$d(n(1 + i't)) \equiv d(n) \pmod{p}.$$

By Dirichlet's Theorem, there exists an arbitrarily large i' for which $1 + i't = p'$ where p' is prime. If we choose such an i' and let $n = p'$, then

$$d((p')^2) \equiv d(p') \pmod{p}$$

which implies

$$3 \equiv 2 \pmod{p},$$

a contradiction. Hence, $d(n)$ is not ultimately periodic, and we are done. ∎

The following theorem is a generalization of Theorem 2.10 to $\mathbb{F}_q(t)$ where q is any prime power.

Theorem 2.11. *Suppose that $q = p^k$ for $k \geq 1$ and some prime p. The Carlitz π $\pi_q(t)$ is transcendental over $\mathbb{F}_q(t)$.*

We end this section by answering the question about when a formal power series is algebraic over two different finite fields $\mathbb{F}_m(X)$ and $\mathbb{F}_n(X)$ for integers m and n. In the language of automatic sequences, this is Cobham's Theorem (Theorem 1.6).

Theorem 2.12 (Cobham, reformulated)**.** *Let $(a_k)_{k\geq 0}$ be a sequence with its values in a finite set $\mathcal{A}$, and $\mathbb{F}_m(X)$ amd $\mathbb{F}_n(X)$ be two finite fields. Let ϕ_m and ϕ_n be injective maps such that that map A to $\mathbb{F}_m(X)$ and $\mathbb{F}_n(X)$, respectively.*

(1) *If m and n are multiplicatively dependent, then $\sum_{k\geq 0} \phi_m(a_k)X^k$ is algebraic over $\mathbb{F}_m(X)$ if and only if $\sum_{k\geq 0} \phi_n(a_k)X^k$ is algebraic over $\mathbb{F}_n(X)$.*

(2) *If m and n are multiplicatively independent and $\sum_{k\geq 0} \phi_\alpha(a_k)X^k$ is algebraic over $\mathbb{F}_\alpha(X)$ for $\alpha \in \{m, n\}$, then $\sum_{k\geq 0} \phi_\alpha(a_k)X^k$ is rational for $\alpha \in \{m, n\}$.*

3. Multidimensional Automatic Sequences

3.1. **Introduction and Formal Power Series Revisited.** As we saw in Chapter 2, automatic sequences were useful in proving algebraicity results on univariate formal power series. If we consider extending automatic sequences to several dimensions, then we would expect to obtain similar algebraicity results on multivariate formal power series. In this section we state the multivariate analogues of many of the definitions/results obtained earlier in Chapter 2.

Definition 3.1. A $[k, l]$-*DFAO* is a 6-tuple $(Q, \Sigma_k, \delta, q_0, \Delta, \tau)$, where

- Q is a finite set called the set of *states*,
- $\Sigma_{k,1}$ is the set of ordered pairs of strings whose elements are in $\{0, 1, 2, \ldots, k-1\} \times \{0, 1, 2, \ldots, l-1\}$
- $\delta : Q \times \Sigma_{k,1} \to Q$ is called the *transition function*,
- $q_0 \in Q$ is called the *starting state*,
- Δ is a finite set called the *output alphabet*,
- $\tau : Q \to \Delta$ is the output function mapping from the set of starting states to the output alphabet.

Definition 3.2. The two-dimensional array $(u_{i,j})_{i,j\geq 0}$ is *[k,l]-automatic* if and only if there exists a $[k, l]$-DFAO $(Q, \Sigma_{k,l}, \delta, \Delta, \tau)$ such that $u_{i,j} = \tau(\delta(q_0, ([i]_k, [j]_l)))$ for all $i, j \geq 0$.

We say that a two-dimensional array that is a $[k, k]$-automatic sequence is *k-automatic*.

Theorem 3.3 (Periodic Indexing)**.** *Let $(s_{m,n})_{m,n\geq 0}$ be a two-dimensional array with values in a finite set such that there exist two integers $a \geq 1$ and $b \geq 1$ for which all the sequences $(s_{am+c,bn+d})_{m,n\geq 0}$ with $c \in [a]$ and $d \in [b]$ are k-automatic for some integer $k \geq 2$. Then $(s_{m,n})_{m,n\geq 0}$ is k-automatic.*

We shift our attention towards formal power series.

Definition 3.4. The ring $K[[X,Y]]$ for some ground field K is the set given by

$$\left\{ \sum_{m,n\geq 0} a_{m,n}X^mY^n : a_{m,n} \in K \right\}.$$

The field $K((X,Y))$ is similarly given by

$$\left\{ \sum_{m\geq -m_0, n\geq -n_0} a_{m,n}X^mY^n : m_0, n_0 \in \mathbb{Z}, a_{m,n} \in K \right\}.$$

Definition 3.5. We say that a bivariate Laurent series $F(X) \in K((X,Y))$ is algebraic over the field of rational functions $K(X)$ if for some integer d there exists polynomials $A_0(X,Y), \ldots, A_d(X,Y)$ with coefficients in K and not all 0 such that

$$\sum_{i=0}^{d} A_i(X,Y)F(X,Y)^i = 0.$$

Now, we state the multivariate analogue of Christol's Theorem.

Theorem 3.6 (Multivariate Christol)**.** *Let p be any prime number. A formal power series $\sum_{n=0}^{\infty} a_{m,n}X^mY^n \in \mathbb{F}_p(X,Y)$ is algebraic over $\mathbb{F}_p(X,Y)$ if and only if $(a_n)_{n\geq 0}$ is p-automatic.*

A more general version of Theorem 3.5 to $\mathbb{F}_q(X)$ is stated in Theorem 3.6, where q is a prime power.

Theorem 3.7. *Let q be any prime power with $q > 1$. A formal power series $\sum_{n=0}^{\infty} a_{m,n}X^mY^n \in \mathbb{F}_q(X,Y)$ is algebraic over $\mathbb{F}_q(X,Y)$ if and only if $(a_n)_{n\geq 0}$ is q-automatic.*

The following theorem is a consequence of Theorem 3.5.

Theorem 3.8. *If a formal power series $F(X,Y)$ with integer coefficients is algebraic over $\mathbb{Q}(X,Y)$, then the coefficients of $F(X,Y)$ modulo p form a p-automatic sequence.*

3.2. Pascal's Triangle modulo d.

In this section we study the two-dimensional array for Pascal's Triangle modulo d, or in other words the 2-dimensional sequence

$$\left(\binom{m}{n} \pmod{d} \right)_{m,n\geq 0}.$$

(Assume that $\binom{m}{n} = 0$ whenever $m < n$.)

Question 3.9. *For which values of $d, k \geq 2$ is the sequence $\left(\binom{m}{n} \pmod{d} \right)_{m,n\geq 0}$ k-automatic?*

In order to answer the above question, we first state the following lemma.

Lemma 3.10. *Let $R(X)$ and $G(X)$ be polynomials in $\mathcal{R}[X]$, where $\mathcal{R}$ is a finite commutative unitary ring, and suppose that there exists $k \geq 2$ such that $R(X^k) = R^k(X)$. Then the sequence $b_{m,n} := [X^m]G(X)R(X)^n$ is k-automatic.*

Theorem 3.11. *The sequence $\left(\binom{m}{n} \pmod{d} \right)_{m,n\geq 0}$ is k-automatic if and only if d and k are powers of the same prime p.*

Proof. Suppose that $d = p^l$ for some prime p and integer $l \geq 1$. If we let

- $\mathcal{R} = \mathbb{Z}/p^l\mathbb{Z}$,
- $R(X) = (1+X)^{p^{l-1}}$, $G(X) = (1+X)^t$ where $0 \leq t \leq p^{l-1} - 1$,
- $k = p$

in Lemma 3.10, then

$$b_{m,n} = [X^m]G(X)R(X)^n = [X^m](1+X)^{p^{l-1}n+t} = \binom{p^{l-1}n+t}{m}.$$

Thus, $\left(\binom{p^{l-1}n+t}{m}\right)_{m,n\geq 0}$ is p-automatic for all $0 \leq t \leq p^{l-1}-1$. By Theorem 3.3,

$$\left(\binom{n}{m}\right)_{m,n\geq 0} = \left(\binom{m}{n}\right)_{m,n\geq 0}$$

is p-automatic, and is thus p^j-automatic for all $j \geq 1$.

Suppose that $d \geq 2$ is not a power of a prime; we will now show that for all $k \geq 2$,

$$\left(\binom{m}{n} \pmod d\right)_{m,n\geq 0}$$

is not k-automatic. We do so by considering two distinct cases for d.

Case 1: d has two distinct odd prime divisors. Let two of these divisors be p_1 and p_2.

Let

$$F(X) = \sum_{i\geq 0} \binom{2i}{i} X^i$$

and note that

$$(1-4X)F(X)^2 - 1 = 0$$

and that $F(X)$ is not a rational function. By Theorem 2.5, $\left(\binom{2n}{n} \pmod p\right)_{n\geq 0}$ is an automatic sequence that is not ultimately periodic for any odd prime p. Suppose that $\left(\binom{m}{n} \pmod d\right)_{m,n\geq 0}$ is k-automatic for some $k \geq 2$. Then, $\left(\binom{2n}{n} \pmod d\right)_{n\geq 0}$ is also k-automatic. Since $p_1|d$, the Projection Theorem [2] implies $\left(\binom{2n}{n} \pmod{p_1}\right)_{n\geq 0}$ is k-automatic.

We also have that $\left(\binom{2n}{n} \pmod{p_1}\right)_{n\geq 0}$ is p_1-automatic, so by Cobham's Theorem p_1 and k are multiplicatively dependent, which further implies that $\frac{\log(p_1)}{\log(k)}$ is rational. Analogously, $\frac{\log(p_2)}{\log(k)}$ is rational, so $\frac{\log(p_1)}{\log(p_2)}$ is also rational, a contradiction.

Case 2: $d = 2^a p^b$ for some odd prime p and integers $a \geq 1$ and $b \geq 1$.

Notice that by the Lagrange Inversion Theorem on $f(x) = x^{-2} - x^{-3}$,

$$G(X) = \sum_{i\geq 0} \frac{\binom{3i}{i}}{2i+1} X^i$$

is a solution to

$$XG^3 - G + 1 = 0.$$

Let

[2]This is a reference to Theorem 1.5.

$$G'(X) = \sum_{i \geq 0} \binom{3i}{i} X^i.$$

Then, $G = G'$ over $\mathbb{F}_2(X)$, so

$$X(G')^3 + G' + 1 = 0$$

over $\mathbb{F}_2(X)$. We now show that G' is not a rational function. Assume to the contrary that $G'(X) = \frac{P(X)}{Q(X)}$ for coprime polynomials $P(X)$ and $Q(X)$ that are in $\mathbb{F}_2[X]$. Then,

$$XP^3 + PQ^2 + Q^3 = 0.$$

Thus, Q divides X. If $Q = 1$, then

$$XP^3 + P + 1 = 0,$$

which is not possible. If $Q = X$, then

$$XP^3 + X^2P + X^3 = 0,$$

which implies that X divides P, a contradiction since P and Q are coprime. Thus, G' is not a rational function. By Theorem 2.5, $\left(\binom{3i}{i} \pmod 2\right)_{i \geq 0}$ is 2-automatic and is not ultimately periodic.

A similar argument using Lagrange inversion on $f(x) = x^{-p} - x^{-p-1}$ yields that $\left(\binom{(p+1)i}{i} \pmod p\right)_{i \geq 0}$ is p-automatic and not ultimately periodic.

Suppose that $\left(\binom{m}{n} \pmod d\right)_{m,n \geq 0}$ is k-automatic for some $k \geq 2$. Then, $\left(\binom{3n}{n} \pmod d\right)_{n \geq 0}$ is also k-automatic. Since $2|d$, the Projection Theorem implies that $\left(\binom{3n}{n} \pmod 2\right)_{n \geq 0}$ is k-automatic. We also have that $\left(\binom{3n}{n} \pmod 2\right)_{n \geq 0}$ is 2-automatic and is not ultimately periodic, so by Cobham's Theorem, 2 and k are multiplicatively dependent, which further implies that k is a power of 2. Using the fact that $\left(\binom{(p+1)i}{i} \pmod p\right)_{i \geq 0}$ is p-automatic and not ultimately periodic, we can similarly find that k is a power of p, a contradiction.

Hence, no value of d that is not a power of an odd prime exists such that there exists some $k \geq 2$ for which $\left(\binom{m}{n} \pmod d\right)_{m,n \geq 0}$ is k-automatic. However, we showed that whenever $d = p^l$ for some prime p and integer $l \geq 1$, $\left(\binom{m}{n} \pmod d\right)_{m,n \geq 0}$ is p-automatic, which completes the proof.

■

References

[AS03] Jean-Paul Allouche and Jeffrey Shallit. *Automatic Sequences: Theory, Applications, Generalizations.* Cambridge University Press, 2003.