Cellular Automata and the Game of Life

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Cellular automata can be used to model various real-world processes. For example, they can model chemical reactions. A notable cellular automaton is John Conway's Game of Life.

Cellular Automata

A *cellular automaton* models a system which consists of objects called *cells*. It has the following characteristics:

- Every cell has a state.
- Every cell exists on a grid.
- Every cell has a neighborhood, which is usually a list of cells adjacent to a specific cell.
- A transition function which dictates how a new state is produced given the current state of a cell.

More formally, a cellular automaton is a tuple $(\mathbb{Z}^n, S, N, f)$, where

- $\mathbb{Z}^n$ with the dimension $n \geq 1$.
- S is a finite set of states.
- N is the neighborhood with $N = (n_1, n_2, ..., n_m) \in \mathbb{Z}^n$.
- $f : S^{|N|} \to S$ is the transition function.

A *pattern* is a configuration of cells on the grid which the cellular automaton is operating on. When the rule is applied, all cells change at the same time. In other words, a configuration (or pattern) c is changed into a configuration c' where

$$c'(n) = f(c(n + n_1), ..., c(n + n_m)).$$

The *global transition function* G is the transformation $c \mapsto c'$.
A *generation* is the unit of time in a cellular automaton. Then, generation 0 is the starting configuration and generation 1 is the pattern obtained after

applying the transition function.
The *orbit* of c is the sequence

$$orb(c) = c, G^1(c), G^2(c), ...$$

where c is the initial configuration and $G^t(c)$ is the configuration at time t. A pattern is the *parent* of a pattern if it produces it after one generation.
A configuration c is

- a *fixed point* if $G(c) = c$.
- *eventually fixed* if $G^n(c) = G^{n+1}(c)$ for some n.
- *periodic* if $G^t(c) = c$ for some t.
- *eventually periodic* if $G^n(c) = G^{n+t}(c)$ for some n.

The smallest t satisfying $G^t(c) = c$ is the *least period* of c.
A cellular automaton is injective, bijective, or surjective if its transition function is injective, bijective, or surjective.

Garden of Eden (GOE)

A configuration c is a *GOE* (*Garden of Eden*) configuration if it has no pre-images. Additionally, a cellular automaton can only have GOE configurations if it is not surjective. In other words, a GOE can have no parents and therefore occurs in generation 0.
An *orphan* is a finite pattern that cannot evolve from another pattern. In other words, a finite pattern without a pre-image is an orphan.
Proposition: Let $c_1, c_2, ...$ be a converging sequence of configurations. Then the sequence $G(c_1), G(c_2), ...$ converges and

$$c = \lim_{i \to \infty} c_i$$

in

$$\lim_{i \to \infty} G(c_i) = G(c).$$

Proof. Let G be a transition function. Let $n_1 \in \mathbb{Z}^n$ and let $k = \max\{k_1, k_2, ..., k_m\}$. Since

$$c = \lim_{i \to \infty} c_i,$$

for every $j = 1, 2, ..., m$ there exists $k_j \in \mathbb{Z}$ with $k_j > 0$ such that

$$c_i(n + n_j) = c(n + n_j) \text{ for all } i \geq k_j.$$

When $k \leq i$,

$$\begin{aligned} G(c_i)(n) &= f(c_i(n + n_1), ..., c_i(n + n_m)) \\ &= f(c(n + n_1), ..., c(n + n_m)) \\ &= G(c)(n). \end{aligned}$$

Then, $G(c_1), G(c_2), ...$ converges to $G(c)$. ∎

Proposition: Every GOE configuration has a subpattern that is an orphan.

Proof. Let c_E be a GOE configuration. Suppose c_E has no subpattern which is an orphan. Let $e_1, e_2, ...$ denote the elements of $\mathbb{Z}^n$ and let

$$D_j = \{e_1, e_2, ...\}$$

for every $j \in \mathbb{Z}$ with $j > 0$. Given the subpattern of c_E with domain D_j is not an orphan, there is a configuration c_j such that $G(c_j)$ agrees with c_E in D_j. Then, the sequence

$$G(c_{E_1}), G(c_{E_2}), ...$$

converges to c_E. Additionally, the sequence

$$c_{E_1}, c_{E_2}, ...$$

has a converging subsequence

$$c_{i_1}, c_{i_2}, ...$$

(where c denotes c_E) with a limit l. Then, the sequence

$$G(c_{i_1}), G(c_{i_2}), ...$$

converges to $G(l)$. However,

$$\lim_{j\to\infty} G(c_{i_j})$$
$$= \lim_{i\to\infty} G(c_i)$$
$$= c_E.$$

Then, $G(l) = c_E$ and therefore c_E is not a GOE. ■

Moore Neighborhood

The *Moore neighborhood* consists of the cells orthogonally or diagonally adjacent to the area being discussed. This is in contrast to the *von Neumann neighborhood*, which consists only of the cells orthogonally adjacent to the area being discussed.

Game of Life

John Conway's *Game of Life* is a cellular automaton. Its universe is an infinite 2-dimensional grid consisting of square cells (like graph paper). The cells have two possible states, *on* (live) or *off* (dead). Each cell interacts with eight adjacent neighbors (a Moore neighborhood). The following transitions are occur:

- Any cell which is live with fewer than two neighbors which are live dies. This is refered to as *underpopulation*.

- Any live cell with more than three live neighbors dies. This is refered to as *overpopulation*.
- Any live cell with two or three live neighbors remains unchanged.
- Any dead cell with exactly three live neighbors will come to life.

The rules meet certain criteria, namely

- There shouldn't be a starting pattern which has a simple proof that the population can grow without limit.
- There should be starting patterns that seem to grow without limit.
- There should be starting patterns that evolve for a while before coming to an end in the following ways:
  - Going away completely.
  - Staying in a stable configuration that remains unchanged.
  - Oscillating in an endless cycle of two or more periods.

Then, there is not an algorithm which can decide if, given the initial pattern and a later pattern, the initial pattern will evolve into the later pattern. A *polyomino* is a finite pattern of orthogonally connected connected cells.
The *R-pentomino* is a polyomino which does not stabilize until generation 1103. This is in contrast to every other polyomino, which will stabilize in at most 10 generations.