RIEMANN-HURWITZ AND RIEMANN-ROCH

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Contents

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0. Introduction

This expository paper is to present what I've learned about Riemann surfaces from [Mir97] for the Euler Circle Algebraic Geometry class. The main purpose is to supply the necessarily background and present the Riemann-Hurwitz Formula as well as the Riemann-Roch Theorem. Here's a summary of contents in each section (mainly following the progresses of [Mir97]):

- In Section 1, we define Riemann surfaces and present a number of examples;
- In Section 2, we define holomorphic and meromorphic functions on Riemann surfaces, as well as orders of meromorphic functions at a point. Meromorphic functions, specifically, deserve special attention for further discussions towards the Riemann-Roch Theorem, so we present them in a separate section;
- In Section 3, we define holomorphic maps between Riemann surfaces as well as multiplicities of holomorphic maps at a point. We will see that this *local* concept relates to a *global* behavior of holomorphic maps, characterized by degrees. Finally, we present the Riemann-Hurwitz Formula, relating the genera of the domain and range of a holomorphic map between compact Riemann surfaces;
- In Section 4, we first introduce 1-forms and 2-forms on Riemann surfaces, which are necessary for defining integration on Riemann surfaces. We then present two important theorems regarding integration: Stoke's Theorem and the Residue Theorem;
- In Section 5, we introduce divisors, which packs local information of Riemann surfaces and maps on them into a single object. We then define the space $L(D)$, which is a measure of meromorphic functions with certain local behaviors on a compact Riemann surface and is the subject of the Riemann-Roch Theorem;
- In Section 6, we introduce algebraic curves, which are compact Riemann surfaces equipped with a decent supply of nontrivial meromorphic functions. We then focus on the Mittag-Leffler Problem, which asks for the existence of meromorphic functions with prescribed local behaviors while being holomorphic everywhere else. The Mittag-Leffler Problem can be measured by the space $H^1(D)$, which we identify with $L^{(1)}(D)$ via Serre Duality. Finally, we present the Riemann-Roch Theorem, which measures the dimension of $L(D)$.

1. Riemann Surfaces

1.1. Preliminaries: Holomorphic Functions on $\mathbb{C}$.

Holomorphic functions on the complex plane are essential objects of one-variable complex analysis. Since they're so important, we will briefly go through them. Holomorphic functions on $\mathbb{C}$ have a wealth of beautiful properties, but we will only reference them when we use the specific results.

Definition 1.1.1 (Holomorphic Functions on $\mathbb{C}$)**.** A complex-valued function $f : \mathbb{C} \to \mathbb{C}$ is said to be *complex-differentiable at a point* $z_0 \in \mathbb{C}$, if its derivative at z_0, defined as the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

exists. f is said to be *holomorphic at z_0* if it is complex-differentiable in a an open subset containing z_0 (which is often called a *neighborhood* of z_0). f is said to be *holomorphic in an open set U* if f is holomorphic at any point of U.

Examples 1.1.2.

(1) Any polynomial function $f(z) = \sum_{k=0}^{n} a_k z^k$ is holomorphic in all of $\mathbb{C}$ with derivative $f'(z) = \sum_{k=1}^{n} k a_k a^{k-1}$.
(2) The function $g(z) = 1/z$ is holomorphic in $\mathbb{C} \setminus \{0\}$ with derivative $g'(z) = -1/z^2$ for $z \neq 0$.
(3) The function $h(z) = |z|$ is complex differentiable only at 0, so it is not holomorphic at any point.

A fundamental characterization of holomorphic functions on $\mathbb{C}$ are the Cauchy-Riemann equations (we only work with $\mathcal{C}^\infty$, or *infinitely differentiable*, functions, so we present a weak version; in particular, any holomorphic function in $V \subset \mathbb{C}$ is $\mathcal{C}^\infty$ on V):

Theorem 1.1.3 (Cauchy-Riemann equations)**.** *A $\mathcal{C}^\infty$ function $f(x + iy) = u(x, y) + iv(x, y)$, where u and v are $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ functions, on some open set $V \subset \mathbb{C}$ is holomorphic in V iff u and v satisfies the Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

1.2. **Charts and Atlases.** Roughly speaking, Riemann surfaces are surfaces on which we may conduct complex analysis. Surfaces locally look like the real plane $\mathbb{R}^2$, and we want Riemann surfaces to locally look like the complex plane $\mathbb{C}$ so we can associate points on the plane with complex variables.

Definition 1.2.1 (Complex charts)**.** Let X be a topological space. A *complex chart* (or simply *chart*) on X is a homeomorphism[1.1] $\phi : U \to V$, where U is an open set in X and V is an open set in $\mathbb{C}$. The open set U is called the *domain* of the complex chart ϕ. The chart ϕ is said to be *centered at* $p \in U$ if $\phi(p) = 0$.

Complex charts gives a local complex coordinate system on its domain, namely $z = \phi(x)$ for $x \in U$.

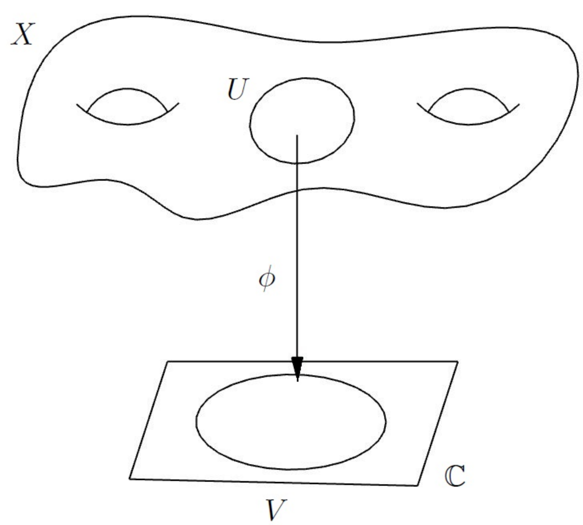

Figure 1. A complex chart ϕ on X

Examples 1.2.2.

(1) Let $X = \mathbb{R}^2$ and U be any open subset of X. The chart $\phi_U : U \to \mathbb{C}$ defined by $\phi_U(x, y) = x + iy$ is a complex chart on $\mathbb{R}^2$. It is centered at 0 iff $0 \in U$.

(2) Again let $X = \mathbb{R}^2$ and U be any open subset of X. The charts
$$\phi_U(x, y) = \frac{x}{1 + \sqrt{x^2 + y^2}} + i\frac{y}{1 + \sqrt{x^2 + y^2}}$$
is also a complex chart on $\mathbb{R}^2$. It is also centered at 0 iff $0 \in U$.

[1.1] A *homeomorphism* is a continuous bijection with a bijective inverse.

(3) Let $\phi : U \to V$ be a complex chart on X and $\psi : V \to W$ be a holomorphic bijection between two open sets of the complex plane. Then the composition $\psi \circ \phi : U \to W$ is a complex chart on X. The composition by ψ can be intuitively thought of as a change of local coordinates; namely, we change from $z = \phi(x)$ to $w = (\psi \circ \phi)(x)$ for $x \in U$.
(4) Given a complex chart $\phi : U \to V$ on X, suppose U_1 is an open subset of U, then $\phi\big|_{U_1} : U_1 \to \phi(U_1)$ is a complex chart on X. This is called a *sub-chart* of ϕ.

We do not want to expect essentially different structures of functions and forms on open sets U when using different charts on U. In other words, we want locally coordinates of any point p to be "similar" when we use different charts containing p. It turns out that we want the following notion for this similarity:

Definition 1.2.3 (Compatibility of charts)**.** Two complex charts $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ on X are said to be *compatible* if either $U_1 \cap U_2 = \emptyset$, or the *transition map*

$$T = \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is holomorphic.

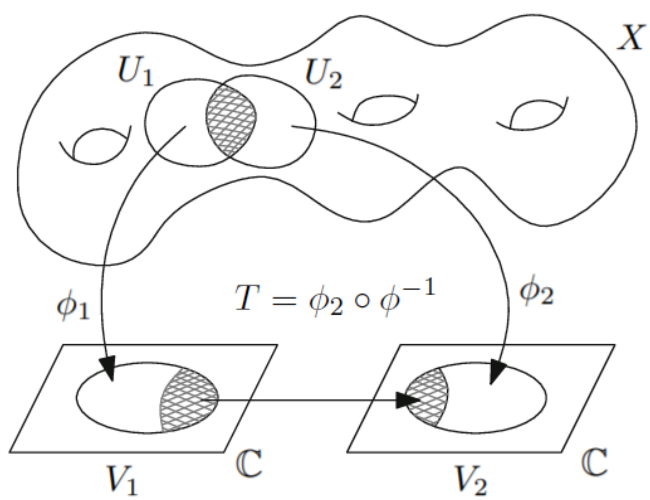

Figure 2. Transition map and compatibility of charts

Examples 1.2.4.

(1) Any two charts in (1) of Examples 1.2.2 are compatible. When $U_1 \cap U_2$ is nonempty, the transition map is the identity on $U_1 \cap U_2$, which is holomorphic.

(2) Any two charts in (2) of Examples 1.2.2 are compatible.
(3) No chart from (1) of Examples 1.2.2 is compatible with any chart from (2), unless their domains are disjoint.
(4) The charts ϕ and $\psi \circ \phi$ from (3) of Examples 1.2.2 are compatible.
(5) Any two sub-charts of a common complex chart are compatible.
(6) Let $\mathbb{S}^2$ denote the unit 2-sphere in $\mathbb{R}^3$, i.e.

$$\mathbb{S}^2 = \{(x, y, w) \in \mathbb{R}^3 : x^2 + y^2 + w^2 = 1\}.$$

The stereographic projection from the north pole $(0, 0, 1)$ to the plane $w = 0$ (viewed as a copy of $\mathbb{C}$) and the projection from $(0, 0, -1)$ followed by a complex conjugation are two compatible charts. Specifically, projection from $(0, 0, 1)$ is the chart $\phi_1 : \mathbb{S}^2 \setminus \{(0, 0, 1)\} \to \mathbb{C}$ defined by

$$\phi_1(x, y, w) = \frac{x}{1-w} + i\frac{y}{1-w}$$

with inverse $\phi_1^{-1} : \mathbb{C} \to \mathbb{S}^2 \setminus \{(0, 0, 1)\}$ given by

$$\phi_1^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$

Similarly, the projection from $(0, 0, -1)$ followed by a complex conjugation is the chart $\phi_2 : \mathbb{S}^2 \setminus \{(0, 0, -1)\} \to \mathbb{C}$ defined by

$$\phi_2(x, y, w) = \frac{x}{1+w} - i\frac{y}{1+w}$$

with inverse

$$\phi_2^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{-2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right).$$

The common domain is $\mathbb{S}^2 \setminus \{(0, 0, \pm 1)\}$, mapped by both ϕ_1 and ϕ_2 homeomorphically onto the open set $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. The transition map $\phi_2 \circ \phi_1^{-1}$ is the map $z \mapsto 1/z$, which is holomorphic on $\mathbb{C}^\times$. $\phi_1 \circ \phi_2^{-1}$ is its inverse and also holomorphic on $\mathbb{C}^\times$, so the two charts are compatible.

Our ultimate goal is to produce complex coordinates (via complex charts) on any given point of X. Moreover, we want these charts to be compatible.

Definition 1.2.5 (Complex atlases)**.** A *complex atlas* (or simply *atlas*) $\mathcal{A}$ on X is a collection

$$\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha\}$$

of pairwise compatible complex charts where $\bigcup_\alpha U_\alpha = X$.

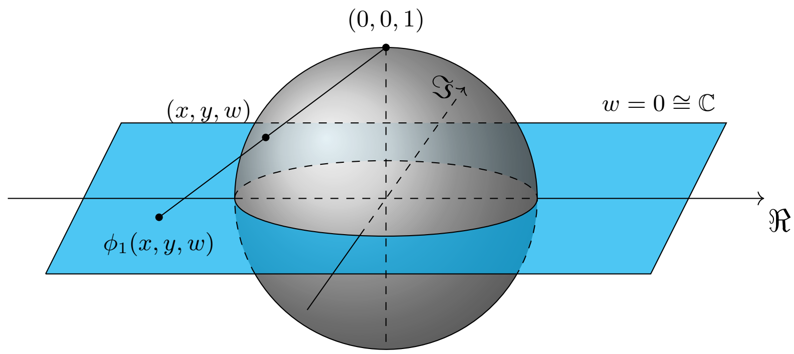

Figure 3. Stereographic projection from the north pole: `tikz` code modified from https://tex.stackexchange.com/questions/538970/how-to-improve-this-stereographic-projection

Sometimes two different atlases give the same local coordinates for X, and we should consider these atlases as essentially the same. The way to do this is to focus on *equivalence classes* of complex atlases on X.

Definition 1.2.6 (Complex structures)**.** Two complex atlases $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if every chart of one is compatible with every chart of the other. This gives an equivalence relation on complex atlases on X. A *complex structure* on X is an equivalence class of complex atlases on X.

Examples 1.2.7.

(1) The collection of complex charts in (1) of Examples 1.2.2 is an atlas on $\mathbb{C}$.
(2) The collection of complex charts in (2) of Examples 1.2.2 is also an atlas on $\mathbb{C}$. This atlas induces a different complex structure from the previous atlas.
(3) If $\mathcal{A} = \{\phi_\alpha : U_\alpha \to V_\alpha\}$ is an atlas on X, and $Y \subset X$ is open, then $\mathcal{A}_Y = \{\phi_\alpha|_{Y \cap U_\alpha} : Y \cap U_\alpha \to \phi_\alpha(Y \cap U_\alpha)\}$ is an atlas on Y.
(4) The collection $\{\phi_1, \phi_2\}$ in (5) of Examples 1.2.4 is an atlas on $\mathbb{S}^2$, since the domains of ϕ_1 and ϕ_2 cover $\mathbb{S}^2$. This atlas induces a complex structure on $\mathbb{S}^2$.

1.3. **Definition and Examples of Riemann Surfaces.** Now we're almost ready to present the definition of Riemann surfaces. We only require two more conditions, mainly to exclude pathological examples: a topological space X is said to be *Hausdorff* if for every pair of distinct points in X there are disjoint neighborhoods of them. X is said to be *second countable* if it admits a countable basis for its topology; we note that X is guaranteed to be second countable if there exists a countable union of open sets which cover X.

Definition 1.3.1 (Riemann surfaces)**.** A *Riemann surface* is a second countable connected Hausdorff topological space X with a complex structure.

Examples 1.3.2.

(1) $\mathbb{C}$ is a Riemann surface with topological properties of $\mathbb{R}^2$ and the complex structure induced by the atlas in (1) of Examples 1.2.7. This Riemann surface is called *the complex plane.*
(2) $\mathbb{S}^2$ is a Riemann surface with its topological properties as a real 2-manifold and the complex structure induced by the atlas in (3) of Examples 1.2.7. This Riemann surface is called the *Riemann Sphere.* The Riemann Sphere is often written as $\mathbb{C}_\infty$ or $\mathbb{C} \cup \infty$, because under either of ϕ_1 or ϕ_2 the image of $\mathbb{S}^2$ is the entire complex plane, with the "point at infinity" ∞ representing the single extra point not in the domain. Unlike $\mathbb{C}$, $\mathbb{C}_\infty$ is a *compact Riemann surface.*
(3) Any connected open subset of a Riemann surface is a Riemann surface, using the atlas in (6) of Examples 1.2.7.

Remark 1.3.3. Note that instead of taking a space with defined topology and impose a complex structure on it, we can often use a given atlas of a topological space X to define a topology on X: assume that we are given a collection of subsets $\{U_\alpha\}$ of set X whose union cover X (the atlas), and a set of bijections $\phi_\alpha : U_\alpha \to V_\alpha$ where V_α is an open subset of $\mathbb{C}$ (the charts). Each V_α inherits a subspace topology from $\mathbb{C}$, so we can declare a subset U of U_α to be open iff $\phi_\alpha(U)$ is open in V_α. This defines a topology on each U_α. Finally, we can define a topology on all of X by declaring a set U to be open in X iff each intersection $U \cap U_\alpha$ is open in U_α. It suffices to check that X is connected and Hausdorff, and each U_α is open in X, which is equivalent to the condition that $\phi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α for all α, β. Note that it is guaranteed that X is second countable if U_α is a countable set.

Using this knowledge we can provide more examples of Riemann surfaces.

(4) Let $\mathbb{CP}^1$ (or simply $\mathbb{P}^1$) be the complex projective line, the set of 1-dimensional subspaces of $\mathbb{C}^2$. Here we denote the span of $(z, w) \in \mathbb{C}^2$ by $(z : w)$.

Let $U_0 = \{(z : w) : z \neq 0\}$ and $U_1 : \{(z : w) : w \neq 0\}$. Note that $U_0 \cup U_1 = \mathbb{P}^1$. We define an atlas on $\mathbb{P}^1$ by $\pi_0 : U_0 \to \mathbb{C}$ by

$$\pi_0(z : w) = \frac{w}{z}$$

and $\pi_1 : U_1 \to \mathbb{C}$ by

$$\pi_1(z : w) = \frac{z}{w}.$$

Both π_0 and π_1 are bijections and $\pi_i(U_0 \cap U_1) = \mathbb{C}^\times$, which is open in $\mathbb{C}$. Therefore we can define the topology on $\mathbb{P}^1$ as remarked above. The transition $\pi_1 \circ \pi_0^{-1}$ is the map $s \mapsto 1/s$, so the two charts are compatible and form an atlas on $\mathbb{P}^1$. Finally, we may check that $\mathbb{P}^1$ is connected and Hausdorff, so $\mathbb{P}^1$ is a Riemann surface. It is called the *complex projective line.* Note that for any $(z : w) \in \mathbb{P}^1$, its image under π_0 is in the closed unit disk

$$\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$$

if $|z| \geq |w|$; otherwise, its image under π_1 is in $\overline{D}$. Thus $\mathbb{P}^1 = \pi_0^{-1}(\overline{D}) \cup \pi_1^{-1}(\overline{D})$, so $\mathbb{P}^1$ is compact since $\overline{D}$ is compact.

(5) Let ω_1 and ω_2 be two complex numbers which are linearly independent over $\mathbb{R}$ (i.e. they're not scalar multiples as vectors in $\mathbb{R}^2$). Define L to be the lattice

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m_1\omega_1 + m_2\omega_2 : m_1, m_2 \in \mathbb{Z}\}.$$

L is a subgroup of the additive group of $\mathbb{C}$, so we may define $X = \mathbb{C}/L$ as the quotient, equipped with the projection group homomorphism $\pi : \mathbb{C} \to X$ and the quotient topology; namely, a set $U \subset X$ is open iff $\pi^{-1}(U)$ is open in $\mathbb{C}$.

L is a discrete subset of $\mathbb{C}$, so there exists $\epsilon > 0$ such that $|\omega| > 2\epsilon$ for every nonzero $\omega \in L$. Fix such an ϵ and fix a point $z_0 \in \mathbb{C}$. The open ϵ-neighborhood $U_{z_0} = V_\epsilon(z_0)$ of z_0 therefore contains at most one point of L. One can check that $\pi\big|_{U_{z_0}} : U_{z_0} \to X$ is injective, and its inverse from $\pi(U_{z_0})$ to U_{z_0} is a complex chart on X; the collection of such charts of all $z_0 \in \mathbb{C}$ forms an atlas on X. One may also check that X is connected, Hausdorff, and second-countable, so X is a Riemann surface. It is indeed compact, since it is also the image of the closed parallelogram

$$P = \{\lambda_1\omega_1 + \lambda_2\omega_2 : \lambda_1, \lambda_2 \in [0, 1]\}$$

under π, and P is compact.

In fact X is topologically a *torus* by considering it as the image of P: the opposite sides of P are identified together and no other identifications are made, which is exactly the identification space of the torus. All these Riemann surfaces (depending on the choice of L) are called *complex tori.*

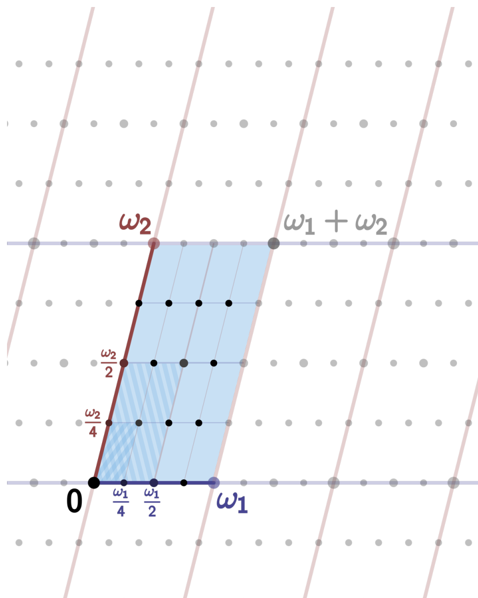

Figure 4. A complex torus, where opposite sides of the parallelogram are identified together

(6) Let $f(z, w) \in \mathbb{C}[z, w]$ be a polynomial over complex variables. Its locus of zeros X is called an *affine plane curve.* We say that f is nonsingular if for all $p \in X$, either the partial derivative $\partial f/\partial z$ or $\partial f/\partial w$ is nonzero at p. The curve X is *smooth* if f is nonsingular.

Theorem 1.3.4. *If $f(z, w)$ is a nonsingular and irreducible polynomial, then its locus or roots X is a Riemann surface. X is called an irreducible smooth affine plane curve.*

Proof. We obtain complex charts on a smooth affine plane curve (not necessarily irreducible) by using the Implicit Function Theorem[1.2].

[1.2] The Implicit Function Theorem states that: let $f(z, w) \in \mathbb{C}[z, w]$, X be its zero locus, and $p = (z_0, w_0) \in X$. Suppose that $\partial f/\partial w(p) \neq 0$. Then there exists a function $g(z)$ defined and holomorphic in a neighborhood of z_0, such that X is equal to the graph $w = g(z)$ near p. Moreover $g' = -\frac{\partial f}{\partial z}/\frac{\partial f}{\partial w}$ near z_0.

Let X be a smooth affine plane curve defined by $f(z, w)$ and $p = (z_0, w_0) \in X$. If $\partial f/\partial w\big|_p \neq 0$, then we find a holomorphic function $g_p(z)$ such that X is the graph $w = g_p(z)$ in a neighborhood U of p. The projection $\pi_z : U \to \mathbb{C}$ mapping $(z, w) \mapsto z$ is a homeomorphism from U to its image in V, which is open in $\mathbb{C}$. If instead $\partial f/\partial z$ is nonzero at p, we make the similar construction and use the other projection π_w mapping $(z, w) \mapsto w$ near p.

We check that any two charts are compatible. If both are projections using π_z or π_w, then the transition map is the identity, which is holomorphic. Now assume that one chart is π_z and the other is π_w. Choose a point $p = (z_0, w_0)$ in the common domain U. Near p, X is locally of the form $w = g(z)$ for some holomorphic function g, and indeed $\pi_w \circ \pi_z^{-1} = g$ on $\pi_z(U)$ near z_0, which is holomorphic.

X is second countable and Hausdorff as a subspace of $\mathbb{C}^2$, and it is connected when f is irreducible. Thus X is a Riemann surface. $\square$

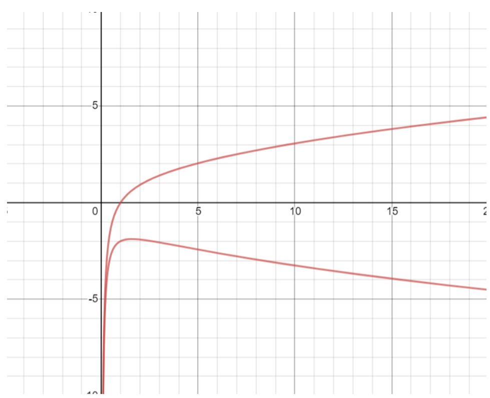

Figure 5. The image of the irreducible smooth affine curve $V(x^3 - (xy + 1)^2)$ in $\mathbb{R}^2$; it's image in $\mathbb{C}^2$ is connected

Finally, Riemann surfaces are 1-dimensional complex manifolds, but we might also think of them topologically as 2-dimensional real manifolds (which are often called *surfaces*), so we immediately inherit much information about their topological structures from the theory of surfaces.

Proposition 1.3.5. *Every Riemann surface is an orientable path-connected smooth 2-dimensional real manifold. Every compact Riemann surface is diffeomorphic to the surface of genus g, for some unique integer g, which is called the genus of the Riemann surface.*

Proof. Every Riemann surface is a 2-dimensional real manifold since it is locally homeomorphic to an open set of $\mathbb{C}$, hence homeomorphic to an open set of $\mathbb{R}^2$. It is smooth since all transition maps are holomorphic, hence $\mathcal{C}^\infty$. in its real variables. It is orientable because holomorphic maps preserves small angles and thus orientations on the complex plane; if we orient the complex plane and induce local orientations on a Riemann surface, the local orientations are well defined and induces a global orientation. Finally, the statement of compact Riemann surfaces follows from the Classification Theorem of Compact Orientable 2-Manifolds. ☐

2. Holomorphic and Meromorphic Functions on Riemann Surfaces

2.1. Definition of Holomorphic and Meromorphic Functions.

Having defined Riemann surfaces as our fundamental objects, we wish to study maps between the Riemann surfaces. As it turns out, we will generalize the notion of holomorphic functions $\mathbb{C} \to \mathbb{C}$ to holomorphic *maps* between Riemann surfaces; this would make a *category* out of Riemann surfaces with the holomorphic maps as the morphisms between them. But before we do so, let's first examine holomorphic *functions* from Riemann surfaces to $\mathbb{C}$. This is easily done since complex charts give a complex coordinate around any point, on which holomorphic functions are readily defined.

Definition 2.1.1 (Holomorphic Functions on Riemann Surfaces)**.** Let X be a Riemann surface, let p be a point of X, and let f be a complex-valued function defined in a neighborhood W of p. We say that f is *holomorphic at* p if there exists a chart $\phi : U \to V$ with $p \in U$, such that the composition $f \circ \phi^{-1} : V \to \mathbb{C}$ is holomorphic at $\phi(p) \in V$. We say that f is *holomorphic in* W if it is holomorphic at every point of W.

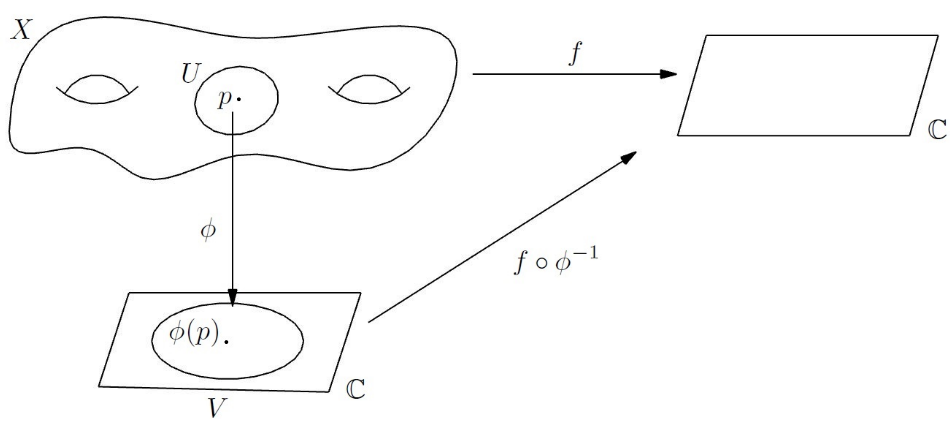

Figure 6. A holomorphic function on X, with $f \circ \phi^{-1}$ holomorphic at $\phi(p) \in V$

We are also interested in the functions that are *almost holomorphic,* that is, functions that are holomorphic away from a discrete set of points:

Definition 2.1.2 (Meromorphic Functions on Riemann Surfaces)**.** Let f be holomorphic in a punctured neighborhood of $p \in X$, i.e. a neighborhood of p excluding p. We say that f is *meromorphic* at p if it is either holomorphic, has a removable singularity at p or has a pole at p. Here, we say f has a removable singularity (resp. pole) at p if there exists a chart $\phi : U \to V$ with $p \in U$ such that the composition $f \circ \phi^{-1}$ has a removable singularity[2.1] (resp. pole[2.2]) at $\phi(p)$.

One can quickly check that the above definitions does not depend on the choice of the chart. Namely, f is holomorphic at p iff $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$ for every chart ϕ whose domain contains p. Similarly, f has a removable singularity (resp. pole) iff $f \circ \phi^{-1}$ has a removable singularity (resp. pole) at $\phi(p)$.

Remark 2.1.3. Since removable singularities of a meromorphic function f are quite irrelevant to the behavior of f, we simply remove them and freely assume that f is holomorphic at z_0. Thus, we assume that meromorphic functions only contain poles as their sole type of singularity in the ensuing discussions.

Examples 2.1.4.

(1) Any complex chart ϕ, considered as a complex-valued function on its domain, is a holomorphic function on its domain, since the composition $\phi \circ \phi^{-1} = \mathrm{id}$.
(2) If f and g are both holomorphic at $p \in X$, then $f \pm g$ and fg are holomorphic at p. If $g(p) \neq 0$, f/g is holomorphic at p.
(3) If f and g are both meromorphic at $p \in X$, then $f \pm g$ and fg are meromorphic at p. If g is not identically zero in a neighborhood of p, then f/g is meromorphic at p.
(4) Let f be a complex-valued function on $\mathbb{C}_\infty$ defined in a neighborhood of ∞. Then f is holomorphic at ∞ iff $f(1/z)$ is holomorphic at $z = 0$, and f is meromorphic at ∞ iff $f(1/z)$ is meromorphic at $z = 0$.

Here is some notation: let $U \subset X$ be an open subset of a Riemann surface X, we denote the set of holomorphic functions on U by $\mathcal{O}_X(U)$ (or simply $\mathcal{O}(U)$) and set of meromorphic functions on U by $\mathcal{M}_X(U)$ (or simply $\mathcal{M}(U)$). We note that both are $\mathbb{C}$-algebras.

[2.1] A function $f : U \setminus \{p\} \to \mathbb{C}$ holomorphic on a punctured neighborhood of $p \in \mathbb{C}$ is said to have a *removable singularity* at p if there is a holomorphic function $g : U \to \mathbb{C}$ that coincides with f on $U \setminus \{p\}$, i.e. we can "remove" the singularity by assigning it an appropriate value.

[2.2] A function f holomorphic on a punctured neighborhood of $p \in \mathbb{C}$ is said to have a *pole* at p if p is a zero of $1/f$.

2.2. **Laurent Series and Order of a Meromorphic Function at a Point.** We can translate the notion of Laurent Series on the complex plane to Riemann surfaces as well, again using local coordinates.

Definition 2.2.1 (Laurent Series about a Point on a Riemann Surface)**.** Let f be holomorphic in a punctured neighborhood of p in a Riemann surface X and let $\phi : U \to V$ be a chart on X with $p \in U$. ϕ gives a local coordinate on U (say $z = \phi(x)$ near p), and $f \circ \phi^{-1}$ is holomorphic in a punctuated neighborhood of $z_0 = \phi(p)$. Thus we may expand $f \circ \phi^{-1}$ in a Laurent Series about z_0 for points z near z_0:

$$(f \circ \phi^{-1})(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

This is called the *Laurent Series for f about p with respect to ϕ.*

Definition 2.2.2 (Order of a Meromorphic Function at a Point)**.** Let f be meromorphic at $p \in X$ and nonconstant near p, whose Laurent series in a local coordinate z is $\sum_n c_n(z - z_0)^n$. The *order of f at p,* denoted by $\operatorname{ord}_p(f)$, is the minimum exponent actually appearing (with nonzero coefficient) in the Laurent series:

$$\operatorname{ord}_p(f) = \min\{n \in \mathbb{Z} : c_n \neq 0\}.$$

Proposition 2.2.3. $\operatorname{ord}_p(f)$ *is well defined. That is, it is independent of the choice of the local coordinate used to define the Laurent series about p.*

Proof. Suppose that $\psi : U' \to V'$ is another chart on X containing p in its domain, giving the local coordinate $w = \psi(x)$ for $x \in U'$. Suppose that $\psi(p) = w_0$. The transformation map $T(w) = \phi \circ \psi^{-1}$ is holomorphic and is the change of coordinates $z = T(w)$.

Since T is in particular bijective, say S is the inverse to T, then $S(T(w)) = w$ for all $w \in U'$. Thus $S'(T(w))T'(w) = 1$, showing that $T'(w)$ is never zero on U'. This means that if we expand

$$z = T(w) = z_0 + \sum_{n\geq 1} a_n (w - w_0)^n,$$

the coefficient a_1 is nonzero.

Suppose now that $f(\phi^{-1}(z)) = c_{n_0}(z - z_0)^{n_0} +$ (higher order terms) is the Laurent series for f about p w.r.t. ϕ, where n_0 is the order of f w.r.t. coordinates z, so $c_{n_0} \neq 0$. The composition of this series with $z - z_0 = \sum_{n\geq 1} a_n(w - w_0)^n$ gives the Laurent series w.r.t. ψ. We immediately see that the lowest degree term in the composition has degree n_0 as well, so n_0 is also the order of f w.r.t. coordinates w. $\square$

The order function behaves nicely with algebraic operations on meromorphic functions: we quickly have

- $\operatorname{ord}_p(fg) = \operatorname{ord}_p(f) + \operatorname{ord}_p(g)$,
- $\operatorname{ord}_p(f/g) = \operatorname{ord}_p(f) - \operatorname{ord}_p(g)$, and
- $\operatorname{ord}_p(1/f) = -\operatorname{ord}_p(f)$

for any functions f and g that are meromorphic at p. Moreover, the order at a point is enough to determine much of the behavior of the meromorphic at this point:

Lemma 2.2.4. *Let f be meromorphic at p. Then f has a pole at p iff $\operatorname{ord}_p(f) < 0$ and f is holomorphic at p iff $\operatorname{ord}_p(f) \geq 0$. f has a zero at p iff $\operatorname{ord}_p(f) > 0$ and f has neither a zero nor a pole at p iff $\operatorname{ord}_p(f) = 0$.*

We say that f has a *zero of order* n at p if $\operatorname{ord}_p(f) = n > 0$ and that f has a *pole of order* n at p if $\operatorname{ord}_p(f) = -n < 0$.

2.3. **Some Theorems about Holomorphic and Meromorphic Functions.** Here are several useful results concerning holomorphic and meromorphic functions inherited immediately from ones in one-variable complex analysis. We get two useful corollaries: the first saying that a meromorphic on a compact Riemann surface is has order 0 away from a finite set, and the second one determines all possible globally holomorphic functions on a compact Riemann surface.

Theorem 2.3.1 (Discreteness of Zeroes and Poles)**.** *Let f be a meromorphic function defined on a connected open set W of a Riemann surface X. If f is not identically zero, then the zeros of poles of f form a discrete subset of W.*

Corollary 2.3.2. *Let f be a meromorphic function on a compact Riemann surface. If f is not identically zero, then f has a finite number of zeroes and poles.*

Theorem 2.3.3 (The Maximum Modulus Theorem)**.** *Let f be holomorphic on a connected open set W of a Riemann surface X. If there is a point $p \in W$ such that $|f(x)| \leq |f(p)|$ for all $x \in W$, then f is constant on W.*

Corollary 2.3.4. *Let X be a compact Riemann surface. If f is holomorphic on all of X, then f is a constant function.*

Proof. Since f is holomorphic, $|f|$ is continuous. Since X is compact, $|f|$ achieves a maximum at point of X. It then follows from the Maximum Modulus Theorem that f is constant. ☐

2.4. **Meromorphic Functions on the Riemann Sphere.** As a concrete example, we will examine the meromorphic functions on the Riemann Sphere. While the previous subsection shows that holomorphic functions on it are not so interesting, this is not the case for meromorphic functions.

Theorem 2.4.1. *Any meromorphic function on the Riemann Sphere is a rational function.*

Proof. Let f be a meromorphic function on $\mathbb{C}_\infty$, then it has finitely many zeroes and poles since $\mathbb{C}_\infty$ is compact. Let $\{\lambda_i\}$ be the set of zeroes and poles of f in $\mathbb{C}$, and $\text{ord}_{\lambda_i}(f) = e_i$. Consider the following rational function

$$r(z) = \prod_i (z - \lambda_i)^{e_i},$$

which has the same zeroes and poles as f to the same orders in $\mathbb{C}$. Let $g(z) = f/r(z)$, then g is a meromorphic function on $\mathbb{C}_\infty$ with no poles or zeros in $\mathbb{C}$, and we wish to prove that g is a constant.

g is holomorphic on all of $\mathbb{C}$, so its Taylor series

$$g(z) = \sum_{n=0}^{\infty} c_n z^n$$

converges everywhere in $\mathbb{C}$. g is also meromorphic at ∞, so

$$g\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} c_n \left(\frac{1}{z}\right)^n$$

is meromorphic at 0. Thus the Taylor series has only finitely many terms; that is, g is a polynomial in z. If, however, g is not constant, then it necessarily has a zero in $\mathbb{C}$, which is a contradiction. Thus g is a constant and $f = gr$ is a rational function like r. □

Note that $\text{ord}_\infty(f) = \text{ord}_\infty(r) = -\sum_i e_i$.

Corollary 2.4.2. *Let f be any meromorphic function on the Riemann Sphere, then*

$$\sum_p \text{ord}_p(f) = 0.$$

As we will see, this is true for *all* meromorphic functions on *any* compact Riemann surfaces.

Similarly, *every meromorphic function on $\mathbb{P}^1$ is a ratio of homogeneous polynomials in z, w of the same degree.* As we will see, this can follow from the fact that $\mathbb{P}^1$ is isomorphic to $\mathbb{C}_\infty$, although one can certainly show it solely using properties of $\mathbb{P}^1$.

3. Holomorphic Maps Between Riemann Surfaces

3.1. **Definition of Holomorphic Maps.** The holomorphic maps between Riemann surfaces are defined again using local coordinates.

Definition 3.1.1 (Holomorphic Maps Between Riemann Surfaces)**.** Let X and Y be Riemann surfaces. A mapping $F : X \to Y$ is *holomorphic at* $p \in X$ if there exists charts $\phi_1 : U_1 \to V_1$ on X with $p \in U_1$ and $\phi_2 : U_2 \to V_2$ on Y with $F(p) \in U_2$ such that the *transition map* $T = \phi_2 \circ F \circ \phi_1^{-1}$ is holomorphic at $\phi_1(p)$. We say F is *holomorphic on* W for an open set $W \subset X$ if F is holomorphic at each point of W. F is a *holomorphic map* if F is holomorphic on all of X.

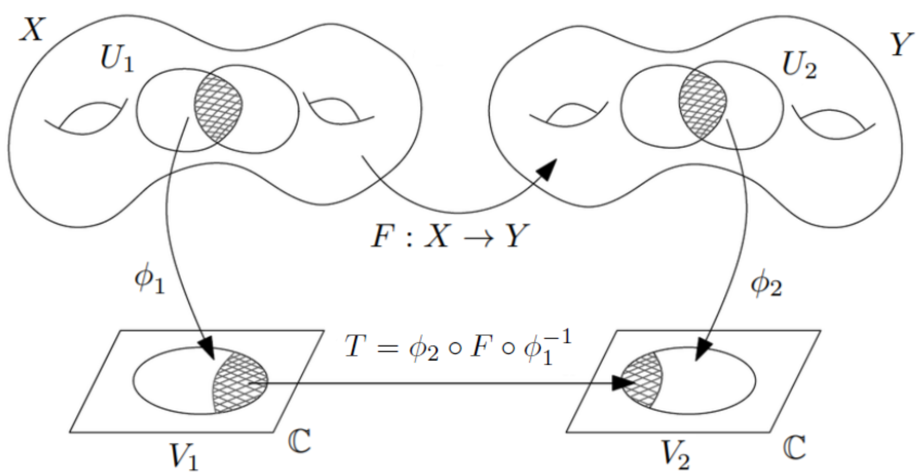

Figure 7. Holomorphic map between Riemann surfaces

Again one can check that the above definitions does not depend on the choice of the charts on X and Y.

Examples 3.1.2.

(1) Any holomorphic function on a Riemann surface X is a holomorphic map from X to $\mathbb{C}$.
(2) The identity mapping $\mathrm{id} : X \to X$ is holomorphic for any Riemann surface X. For any $p \in X$, we find chart $\phi : U \to V$ where $p \in U$ and thus $\phi \circ \mathrm{id} \circ \phi^{-1}$ is the identity map on U, so id is holomorphic at p.
(3) The composition of holomorphic maps $F : X \to Y$ and $G : Y \to Z$ is a holomorphic map $G \circ F : X \to Z$.
(4) The composition of a holomorphic map $F : X \to Y$ with a holomorphic function g on $W \subset Y$ is a holomorphic function $g \circ F$ on $F^{-1}(W)$.

(5) The composition of a holomorphic map $F : X \to Y$ with a meromorphic function g on $W \subset Y$ is a meromorphic function $g \circ F$ on $F^{-1}(W)$, when the image $F(X)$ is not a subset of the poles of g.

Remark 3.1.3. The previous two examples states, if $F : X \to Y$ is a holomorphic map between Riemann surfaces, then for every open set $W \subset Y$, F induces $\mathbb{C}$-algebra homomorphisms

$$F^* : \mathcal{O}_Y(W) \to \mathcal{O}_X(F^{-1}(W))$$

and

$$F^* : \mathcal{M}_Y(W) \to \mathcal{M}_X(F^{-1}(W))$$

given by $F^*(g) = g \circ F$. Note the similarity to induced homomorphisms on coordinate rings by morphisms between algebraic sets. Indeed, we have a *contravariant functor* from Riemann surfaces to rings.

(6) The map $F : \mathbb{P}^1 \to \mathbb{C}_\infty$ defined by

$$F(z : w) = \left(\frac{2\,\mathrm{Re}(z\overline{w})}{|z|^2 + |w|^2}, \frac{2\,\mathrm{Im}(z\overline{w})}{|z|^2 + |w|^2}, \frac{|z|^2 - |w|^2}{|z|^2 + |w|^2} \right) \in \mathbb{S}^2$$

is a holomorphic map from the complex projective line to the Riemann Sphere. F is holomorphic on $U_0 = \{(z : w) : z \neq 0\}$ using the complex charts π_0[3.1] on $\mathbb{P}^1$ and ϕ_2[3.2] on $\mathbb{C}_\infty$. The transition map $T = \phi_2 \circ F \circ \pi_0^{-1}$ is computed to be the identity. Similarly, F is holomorphic on $U_1 = \{(z : w) : w \neq 0\}$ and thus holomorphic on all of $\mathbb{P}^1$.

(7) The map $G : \mathbb{C}_\infty \to \mathbb{P}^1$ defined by

$$G(x, y, w) = \begin{cases} (x + yi : 1 - w) & \text{if } w \neq 1 \\ (1 : 0) & \text{if } w = 1 \end{cases}$$

is a holomorphic map from the Riemann Sphere to the complex projective line. G is holomorphic on $\mathbb{C}_\infty \setminus \{(0, 0, 1)\}$ using the complex charts ϕ_1 on $\mathbb{C}_\infty$ and π_1 on $\mathbb{P}^1$. The transition map $T = \pi_1 \circ G \circ \phi_1^{-1}$ is the identity. Similarly, G is holomorphic on $\mathbb{C}_\infty \setminus \{0, 0, -1\}$ using the complex charts ϕ_2 on $\mathbb{C}_\infty$ and π_0 on $\mathbb{P}^1$. The computation of the transition map $T = \pi_0 \circ G \circ \phi_2^{-1}$

[3.1] See (4) of Examples 1.3.2.

[3.2] See (6) of Examples 1.2.4.

involves casework, but it again turns out to be the identity:

$$\begin{aligned} T(\phi_2(x, y, w)) &= \pi_0(G(x, y, w)) \\ &= \begin{cases} \pi_0(x + yi : 1 - w) & \text{if } w \neq 1 \\ \pi_0(1 : 0) & \text{if } w = 1 \end{cases} \\ &= \begin{cases} (1 - w)/(x + yi) & \text{if } w \neq 1 \\ 0 & \text{if } w = 1 \end{cases} \\ &= \frac{x - yi}{1 + w} \\ &= \phi_2(x, y, w). \end{aligned}$$

With the notion of holomorphic maps it is easy to define isomorphism on Riemann surfaces.

Definition 3.1.4 (Isomorphism of Riemann Surfaces)**.** An *isomorphism* between Riemann surfaces is a bijective holomorphic map $F : X \to Y$ whose inverse is also holomorphic.

Example 3.1.5. The holomorphic maps $F : \mathbb{P}^1 \to \mathbb{C}_\infty$ and $G : \mathbb{C}_\infty \to \mathbb{P}^1$ described above are inverses, thus they are isomorphisms between the complex projective line $\mathbb{P}^1$ is isomorphic to the Riemann Sphere $\mathbb{C}_\infty$.

3.2. **Discreteness of Preimages of a Holomorphic Map.** We have seen that given a meromorphic function f on a Riemann surface X, the set of zeroes and poles of f is a discrete set and is finite when X is compact. This is a nice property since finiteness allows us to easily keep track of the "bad points" when we make a statement that applies to most of the "good points" on X. It also enables us to put together finite formal sum involving points, which we will see in the case of divisors. Therefore, we want to get a similar result for holomorphic maps between Riemann surfaces, this time concerning *preimages*. First, we inherit the following theorem from one-variable complex analysis.

Theorem 3.2.1 (Open Mapping Theorem)**.** *Let $F : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. Then F is an open mapping, i.e. F sends open subsets of X to open subsets of Y.*

Proposition 3.2.2. *Let X be a compact Riemann surface, and let $F : X \to Y$ be a nonconstant holomorphic map. Then Y is compact and X is onto.*

Proof. We shall proof that $F(X)$ is both open and closed in Y. $F(X)$ is open by the Open Mapping Theorem. $F(X)$ is compact as it is the image of X, which is compact, and is hence closed in Y since Y is Hausdorff. Finally, $F(X)$ is nonempty, so it must be all of Y. □

Theorem 3.2.3 (Discreteness of Preimages)**.** *Let $F : X \to Y$ be a nonconstant holomorphic map between Riemann surfaces. Then for every $y \in Y$, the preimage $F^{-1}(y)$ is a discrete subset of X. In particular, if X is compact, then $F^{-1}(y)$ is a nonempty finite set.*

Proof. For $y \in Y$ and $x \in F^{-1}(y)$, choose local coordinates w centered at x and z centered at y. Then the map F, written in these local coordinates, is a nonconstant holomorphic function $z = g(w)$ with $g(0) = 0$. Since zeroes of nonconstant holomorphic functions are discrete, we can select a neighborhood U of x such that x is the only preimage of y in U. Thus $F^{-1}(y)$ is a discrete subset of X. If X is compact, then $F^{-1}(y)$ is finite; it is nonempty since F is onto by the previous proposition. □

3.3. **Multiplicity at a Point of a Holomorphic Map.** Before studying the global behaviors of holomorphic maps, we first investigate a local behavior, which would be of significance shortly. Essentially, the following proposition says that every holomorphic map in local coordinates look like a power map, and the degree of this power map is unique and independent of the local coordinates we choose.

Proposition 3.3.1 (Local Normal Form)**.** *Let $F : X \to Y$ be a non-constant holomorphic map defined at $p \in X$. Then there is a unique integer $m \geq 1$ such that: for every chart $\phi_2 : U_2 \to V_2$ on Y centered at $F(p)$, there exists a chart $\phi_1 : U_1 \to V_1$ on X centered at p such that $T = \phi_2 \circ F \circ \phi_1^{-1}$ is the power map $z \mapsto z^m$.*

Proof. Fix the chart ϕ_2 on Y centered at $F(p)$, and choose any chart $\psi : U \to V$ on X centered at p. Then the Taylor series of the transition map $T = \phi_2 \circ F \circ \psi^{-1}$ has the form

$$T(w) = \sum_{i=m}^{\infty} c_i w^i$$

where $c_m \neq 0$ and $m \geq 1$ since $T(0) = 0$. Thus we have $T(w) = w^m S(w)$ where $S(w)$ is a holomorphic function at $w = 0$ and $S(0) \neq 0$. There exists a function $R(w)$ holomorphic near 0 with $R(0) \neq 0$ such that $R(w)^m = S(w)$ so $T(w) = (wR(w))^m$. Let $\eta(w) = wR(w)$, then $\eta'(0) = R(0) \neq 0$ so η is invertible and holomorphic near 0.

I claim that $\phi_1 = \eta \circ \psi$ is the desired chart on X. It is a chart centered at p since composition by η is a coordinate change. We have a new coordinate z under ϕ_1 near p and it is related to w by $z = \eta(w) = wR(w)$. Thus,

$$\begin{aligned}(\phi_2 \circ F \circ \phi_1^{-1})(z) &= (\phi_2 \circ F \circ \psi^{-1} \circ \eta^{-1})(z) \\ &= T(\eta^{-1}(z)) \\ &= T(w) \\ &= (wR(w))^m \\ &= z^m.\end{aligned}$$

The integer m is unique since, given a point near $F(p)$ in Y, it has exactly m preimages near p in X. Thus m is a topological feature of the map F and is independent of the choice of local coordinates. □

Definition 3.3.2 (Multiplicity)**.** The unique integer m in the setting above is called the *multiplicity* of F at p and is denoted $\text{mult}_p(F)$.

Note that $\text{mult}_p(F) \geq 1$ always. Also, notice in the proof we have discovered that if z and w are any local coordinates centered at p and

$F(p)$, respectively, then the lowest term in the power series expansion of the transition map always has degree $\mathrm{mult}_p(F)$. Thus we have a relationship between the multiplicity of a holomorphic map and the order of its derivative (which is at least meromorphic).

Lemma 3.3.3. *Let $F : X \to Y$ be a nonconstant holomorphic map defined at $p \in X$. Take local coordinates z near p and w near $F(p)$, and say that p is taken to z_0 and $F(p)$ to w_0. In terms of these coordinates, F has the form $w = T(z)$ where T is the transition map: it is holomorphic and $w_0 = T(z_0)$. Then we have the following relation:*

$$\mathrm{mult}_p(F) = 1 + \mathrm{ord}_{z_0}(dT/dz).$$

In particular, the multiplicity is the exponent of the lowest positive term of the power series for T about $z = z_0$.

Proof. Note that $z - z_0$ and $w - w_0$ are local coordinates centered at p and $F(p)$, so $\mathrm{mult}_p(F)$ is the degree of the lowest term appearing in the power series of $w - w_0 = T(z) - T(z_0)$. A quick check of the power series shows that this degree is $1 + \mathrm{ord}_{z_0}(dT/dz)$. $\square$

Thus, since $\mathrm{mult}_p(F) \geq 2$ implies that dT/dz has strictly positive order at z_0, i.e. it has a zero at z_0, by Lemma 2.2.4. *The points p where F has multiplicity at least two is thus a discrete set.* These are "bad points" where easy conclusions about global behaviors fail and thus require additional attention.

Definition 3.3.4 (Ramification and Branch Points)**.** Let $F : X \to Y$ be a nonconstant holomorphic map. A point $p \in X$ is a *ramification point* for F if $\mathrm{mult}_p(F) \geq 2$. A point $y \in Y$ is a *branch point* for F if it is the image of a ramification point for F.

We sense that notion of the order (defined for a meromorphic function) is defined very similarly to that of the multiplicity (defined for a holomorphic map). In fact, they are related more than just via the transition map. We will see this once we can associate a meromorphic function to a reasonable holomorphic map.

Proposition 3.3.5. *Let f be a meromorphic function on a Riemann surface X. We define a function $F : X \to \mathbb{C}_\infty$ by*

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is not a pole of } f \\ \infty & \text{if } x \text{ is a pole of } f. \end{cases}$$

Then F is a holomorphic map between the two Riemann surfaces. In fact, the operation defines a bijection between the set of meromorphic functions on X and the set of holomorphic functions $X \to \mathbb{C}_\infty$ which are not identically ∞.

Proposition 3.3.6. *With the setting of the previous proposition, and assuming that f is not constant, for any $p \in X$, $\operatorname{mult}_p(F)$ and $\operatorname{ord}_p(f)$ are related in the following ways:*

(1) If p is a zero of f, then $\operatorname{mult}_p(F) = \operatorname{ord}_p(f)$.
(2) If p is a pole of f, then $\operatorname{mult}_p(F) = -\operatorname{ord}_p(f)$.
(3) If p is neither a zero nor a pole of f, then $\operatorname{mult}_p(F) = \operatorname{ord}_p(f - f(p))$.

Proof. If p is not a pole of f, then the function $f - f(p)$ has a zero at p. By Lemma 3.3.3, we see that $\operatorname{mult}_p(F) = \operatorname{ord}_p(f - f(p))$; specifically, f has the power series

$$f(z) = f(p) + \sum_{i=\operatorname{mult}_p(F)}^{\infty} (z-p)^i$$

near p, so $f - f(p)$ has order $\operatorname{mult}_p(F)$. In particular, if $f(p) = 0$, then $\operatorname{mult}_p(F) = \operatorname{ord}_p(f)$. If p is a pole of f, then p is a zero of $1/f$. Therefore $\operatorname{mult}_p(F) = \operatorname{ord}_p(1/f) = -\operatorname{ord}_p(f)$. □

3.4. **Degree of a Holomorphic Map.** As said, multiplicity of a holomorphic map is a local behavior, but it relates to the global behavior of a holomorphic map by the following proposition.

Proposition 3.4.1. *Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. For each $y \in Y$, the sum of the multiplicities of F at the preimages of y, i.e.*

$$\sum_{p \in F^{-1}(y)} \mathrm{mult}_p(F),$$

is a constant, independent of y.

Proof. The idea is to show that the function $y \mapsto d_y(F)$ is a locally constant function from Y to $\mathbb{Z}$. Since Y is connected, any locally constant map is globally constant, and we will be done.

First consider the power map $f : D \to D$ given by $z \mapsto z^m$, where $D = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc in $\mathbb{C}$. The map f is holomorphic with the only ramification point being 0. Splitting into cases of $w = 0$ (w has one preimage with multiplicity m) and $w \neq 0$ (w has m preimages each with multiplicity 1), we see that the sum of multiplicities of the preimage is constantly m.

Fix $y \in Y$. We want to show that for any nonconstant holomorphic map F, it is locally a disjoint union of these power maps on the neighborhoods of preimages of y. This is where apply Local Normal Form: let $F^{-1}(y) = \{x_1, \cdots, x_n\}$ and choose a local coordinate w centered at y in Y. By the Local Normal Form Proposition, we may choose coordinates $\{z_i\}$ with z_i centered at x_i for each i, such that in a neighborhood of x_i F is the power map $w = z_i^{m_i}$.

It suffices to check that, for points near y, they have no other preimages which are not in the neighborhoods of the x_i's. Assume for contradiction that we may find points arbitrarily close to y with some preimages not in any of the neighborhoods of x_i's. This is equivalent to saying that we may find a sequence of points $\{p_i\}$ of X, none of which lie in any of the neighborhoods of x_i's, such that the images of these points converge to y. Since X is compact, we have a convergent subsequence $\{p_{n_i}\}$ which converges to a point $x \in X$ and the images $\{F(p_{n_i})\}$ converges to y. Since F is continuous, we have $F(x) = y$, so x is some x_i, which is a contradiction since the p_{n_i}'s are not in any of the neighborhoods of x_i's. □

Definition 3.4.2 (Degree of a Holomorphic Map)**.** The constant sum in the previous proposition is a property of the map F itself and is called the *degree* of F, denoted $\deg(F)$.

Corollary 3.4.3. *A holomorphic map between compact Riemann surfaces is an isomorphism iff it has degree one.*

Proof. Note that a degree-one map is injective, since any point in the range cannot have more than one preimage, and surjective, since any point in the range has a preimage. The inverse is automatically holomorphic. □

Remark 3.4.4. Note that if $q \in Y$ is not a branch point, then $\operatorname{mult}_p(F) = 1$ for all $p \in F^{-1}(q)$, so q has precisely $\deg(F)$ preimages. In fact, if we delete the branch points of F in Y as well as their preimages in X, we obtain a *covering map* $\pi : M \to M'$ between 2-manifolds: every point of the target M' has an open neighborhood $V \subset M'$ such that $\pi^{-1}(M')$ breaks into a disjoint union of open sets $U_i \subset M$, each sent homeomorphically onto V by π. Thus, the map F itself is a *branched covering*, a covering map away from finitely many "bad points" (the branch points).

The with the degree as a tool, and having understood the relation between the order of a meromorphic function and the multiplicity of the induced holomorphic map onto $\mathbb{C}_\infty$, we can prove the following important result, which would again show up in our discussion of divisors:

Proposition 3.4.5. *Let f be a nonconstant meromorphic function on a compact Riemann surface X. Then*

$$\sum_{p \in X} \operatorname{ord}_p(f) = 0.$$

Proof. Let $F : X \to \mathbb{C}_\infty$ be the associated holomorphic map to the Riemann Sphere. Let $\{x_i\} = F^{-1}(0)$ and $\{y_j\} = F^{-1}(\infty)$. Then we have

$$\sum_i \operatorname{mult}_{x_i}(F) = \sum_j \operatorname{mult}_{y_j}(F),$$

since both are both $\deg(F)$. The only points of X where $\operatorname{ord}_p(f) \neq 0$ are the zeroes and poles of f, which are exactly the x_i's and y_j's, thus

$$\begin{aligned}\sum_p \operatorname{ord}_p(f) &= \sum_i \operatorname{ord}_{x_i}(f) + \sum_j \operatorname{ord}_{y_j}(f) \\ &= \sum_i \operatorname{mult}_{x_i}(F) - \sum_j \operatorname{mult}_{y_j}(F) \\ &= 0,\end{aligned}$$

where in the second-to-last equality we've used Proposition 3.3.6: $\operatorname{mult}_{x_i}(F) = \operatorname{ord}_{x_i}(f)$ for each i and $\operatorname{mult}_{y_j}(F) = -\operatorname{ord}_{y_j}(f)$ for each j. □

3.5. **The Euler Characteristic.** The Euler characteristic of a compact Riemann surface is inherited from that of compact 2-manifolds.

Definition 3.5.1 (Euler Characteristic)**.** Let S be a compact 2-manifold (possibly with boundary), and suppose that a triangulation[3.3] of S is given, with v vertices, e edges, and t triangles. Then the *Euler characteristic* of S (w.r.t. this triangulation) is given by $\chi(S) = v - e + t$.

The main result for compact Riemann surfaces is that they can be triangulated. The main result for Euler characteristic is that it's independent of the choice of triangulation.

Proposition 3.5.2. *The Euler characteristic is independent of the choice of triangulation. For a compact orientable 2-manifold without boundary of genus g, the Euler characteristic is $2 - 2g$.*

The usually argument for a proof is to show that the Euler characteristic is unchanged when one *refines* a triangulation, i.e. further "subdivide" the existing triangles by some *elementary refinements*, and that any two triangulations have a common refinement. One then identifies the Euler characteristic of a genus-g 2-manifold with $2 - 2g$.

[3.3] A *triangulation* of a compact 2-manifold is a decomposition of S into closed subsets, each homeomorphic to a triangle, such that any two of these subsets are disjoint, meet only at a single vertex, or meet only along a single edge.

3.6. **The Riemann-Hurwitz Formula.** Here we introduce our first major result, relating, for a holomorphic map F between compact Riemann surfaces, the genera of its domain and range (which are independent of F) with the degree (the "global behavior" of F) and ramification (the "bad points") of F.

First we examine the case where $F : X \to Y$ is any covering map between compact Riemann surfaces. Recall that a covering map $\pi : M \to M'$ between manifolds is such that every point of its target M' has an open neighborhood $V \subset M'$ such that $\pi^{-1}(V)$ breaks into a disjoint union of open sets $U_i \subset M$, each sent homeomorphically onto V by π. We first see that, let $n \in \mathbb{Z}_{\geq 0}$, then the set of points in M with precisely n preimages is both open and close in M, thus it is either empty or all of M. This set is exactly all of M for a unique n, which we call the *degree* of the covering π, denoted $\deg(\pi)$. This is somehow compatible with the notion of degree of holomorphic functions due to Remark 3.4.4.

We will use the following lemma from the theory of covering spaces:

Lemma 3.6.1. *Let $\pi : M \to M'$ be a covering between manifolds, $p_0 \in M$, $p_0' \in \pi^{-1}(p_0)$, and $g : [0,1] \to M$ a path*[3.4] *with $g(0) = p_0$. Then g can be lifted to a unique path $g' : [0,1] \to M'$ with $g'(0) = p_0'$, i.e., that*

$$\pi \circ g' = g.$$

Proposition 3.6.2. *Let $F : X \to Y$ be a covering between triangulable 2-manifolds. Then $\chi(X) = \deg(\pi)\chi(Y)$.*

Proof. Given a triangulation of Y with v vertices, e edges, and v vertices, we view it as a collection of path on Y. Lift each path uniquely to a path on X via π, then we have a triangulation on X with $\deg(\pi)v$ vertices, $\deg(\pi)e$ edges, and $\deg(\pi)v$ vertices. The result thus follows. ☐

As we've noted, a general holomorphic map is a *ramified covering*, so we must slightly adjust the identity above to account for the differences, which the Riemann-Hurwitz Formula does by counting carefully at ramification points.

The Riemann-Hurwitz Formula. *Let $F : X \to Y$ be a nonconstant holomorphic map between compact Riemann surfaces. Then*

$$\chi(X) = \deg(F)\chi(Y) - \sum_{p \in X}[\mathrm{mult}_p(F) - 1],$$

[3.4] A *path* on space X is a continuous function $g : [0,1] \to X$.

or, equivalently,

$$2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{p \in X} [\mathrm{mult}_p(F) - 1].$$

Proof. First we note that the sum is a finite sum since X is compact and the set of ramification points of F is finite.

Take a triangulation of Y with v vertices, e edges, and t triangles such that each branch point of F is a vertex. We can similarly lift this triangulation back to X except for at branch points in Y. However, for a branch point $q \in Y$ and ramification point $p_0 \in F^{-1}(q)$, we may select local coordinates centered at p_0 and q under which F has the form $z \mapsto z^m$, where $m = \mathrm{mult}_p(F)$. Now for any edge on Y viewed as a path on Y starting at q, it is a function $h : [0, 1] \to \mathbb{C}$ under these local coordinates. We take all m-th degree roots of h as lifted paths of g on X starting at p_0. One can show that we have subsequently obtained a triangulation of X by lifting.

Now we count the number of vertices, edges, and triangles of the triangulation on X. Call these values v', e' and t', respectively. Similar to the argument above, we have $t' = \deg(F)t$ and $e' = \deg(F)e$. Vertices are where we need to be careful, and multiplicity comes in as they characterize what happens at ramification points: for any vertex $q \in Y$,

$$\begin{aligned} |F^{-1}(q)| &= \sum_{p \in F^{-1}(q)} 1 \\ &= \deg(F) + \sum_{p \in F^{-1}(q)} [1 - \mathrm{mult}_p(F)]. \end{aligned}$$

Now we can calculate

$$\begin{aligned} v' &= \sum_{\text{vertex } q \text{ of } Y} \left(\deg(F) + \sum_{p \in F^{-1}(q)} [1 - \mathrm{mult}_p(F)] \right) \\ &= \deg(F)v - \sum_{\text{vertex } q \text{ of } Y} \sum_{p \in F^{-1}(q)} [\mathrm{mult}_p(F) - 1] \\ &= \deg(F)v - \sum_{\text{vertex } p \text{ of } X} [\mathrm{mult}_p(F) - 1]. \end{aligned}$$

Finally, putting everything together,

$$\begin{aligned}\chi(X) &= v' - e' + t' \\ &= \deg(F)v - \sum_{\text{vertex } p \text{ of } X} [\mathrm{mult}_p(F) - 1] - \deg(F)e + \deg(F)t \\ &= \deg(F)\chi(Y) - \sum_{\text{vertex } p \text{ of } X} [\mathrm{mult}_p(F) - 1] \\ &= \deg(F)\chi(Y) - \sum_{p \in X} [\mathrm{mult}_p(F) - 1],\end{aligned}$$

where the last equality holds because every ramification point of F is a vertex of X so that $\mathrm{mult}_p(F) = 1$ for any non-vertex of X. □

4. Integration on Riemann Surfaces

4.1. **Holomorphic and Meromorphic 1-Forms.** For the purpose of this paper, the main objective of this section is to prove the Residue Theorem. However, the proof would require Stoke's Theorem on Riemann surfaces, which would require a thorough look into integration theory on Riemann surfaces.

First, we need to define objects to integrate. These objects are called *forms*. Again, the usual approach to define objects on Riemann surfaces is to first define them on open subsets of $\mathbb{C}$, which can be carried over via charts.

Definition 4.1.1 (Holomorphic 1-Forms on $\mathbb{C}$)**.** A *holomorphic 1-form* on an open set $V \subset \mathbb{C}$ is an expression ω of the form

$$\omega = f(z)\,\mathrm{d}z,$$

(note that we write formally $\mathrm{d}z$ instead of dz) where f is a holomorphic function on V. We say that ω is a holomorphic 1-form *in the coordinate* z.

Now we wish to associate holomorphic 1-forms ω_ϕ for each chart $\phi : U \to V$ on a Riemann surface X; however, as we've seen in the example of complex charts, we want to pose some sort of *compatibility condition* on the 1-forms.

Definition 4.1.2 (Transformation of 1-Forms)**.** Suppose that $\omega_1 = f(z)\,\mathrm{d}z$ and $\omega_2 = g(w)\,\mathrm{d}w$ are holomorphic 1-forms defined on open sets V_1, V_2 of $\mathbb{C}$, respectively, and let $z = T(w)$ be the holomorphic transition map from V_2 to V_1. We say that ω_1 *transforms to* ω_2 *under* T if

$$g(w) = f(T(w))T'(w).$$

Note that, however, $T : V_2 \to V_1$ goes in the opposite direction of the transformation.

The above definition is made exactly so that

$$g(w)\,\mathrm{d}w = f(T(w))T'(w)\,\mathrm{d}w = f(z)\,\mathrm{d}z$$

when one sets $\mathrm{d}z = T'(w)\,\mathrm{d}w$. Now we're ready to define a holomorphic 1-form on a Riemann surface.

Definition 4.1.3 (Holomorphic 1-Forms on Riemann Surfaces)**.** Let X be a Riemann surface. A *holomorphic 1-form* ω on X is a collection of holomorphic 1-forms $\{\omega_\phi\}$ on $\mathbb{C}$, one for each chart $\phi : U_\phi \to V_\phi$ and each ω_ϕ is on V_ϕ. Moreover, if two charts ϕ_1, ϕ_2 have overlapping domains, then their associated 1-forms ω_{ϕ_1} transforms to ω_{ϕ_2} under the change of coordinates $T = \phi_1 \circ \phi_2^{-1}$.

Remark 4.1.4. The definition requires that we give a holomorphic 1-form for each chart of X, but it suffices to do so for an atlas $\mathcal{A}$ of X. Indeed, if ψ is a chart of X not in $\mathcal{A}$, we find a point p in its domain and a chart ϕ in $\mathcal{A}$ containing p in its domain. Say ψ has local variable w and ϕ has local variable z, and say $\omega_\phi = f(z)\,\mathrm{d}z$ is the holomorphic 1-form associated with ϕ. Then we simply define $\omega_\psi = f(T(w))T'(w)\,\mathrm{d}w$, where $T = \phi \circ \psi^{-1}$.

We can define *meromorphic 1-forms* on Riemann surfaces using the exact same approach, only with $f(z)$ in Definition 4.1.1 being a meromorphic (instead of holomorphic) function on V. The set of holomorphic 1-forms on a open subset $U \subset X$ is denoted by $\Omega^1(U)$ and the set of meromorphic 1-forms is denoted by $\mathcal{M}^{(1)}(U)$. Both are modules over $\mathcal{O}(U)$, the set of holomorphic functions on U.

Example 4.1.5. We define a meromorphic 1-form ω on the Riemann Sphere $\mathbb{C}_\infty$ using charts from (7) of Example 1.2.4; we mandate that the south pole is 0 and the north pole is ∞. We define $\omega = \mathrm{d}z$ in ϕ_1, the chart whose domain is near 0. Recall that the transition map $T(w) = \phi_2 \circ \phi_1^{-1}$ is $T(w) = 1/w$, thus we must define

$$\omega = f(T(w))T'(w)\,\mathrm{d}w = -1/w^2\,\mathrm{d}w$$

near ∞. This gives a meromorphic 1-form on $\mathbb{C}_\infty$.

Remark 4.1.6. As we've seen in the previous example, in the case of meromorphic 1-forms, it's most convenient to define ω_ϕ for *a single chart* ϕ and then extend it to a global meromorphic 1-form. The data on the single chart ϕ is sufficient to determine the global 1-form, since if two meromorphic functions agree on an open set, they must be identical[4.1].

However, it's not clear that such an ω exists. For example, the meromorphic 1-form $e^z\,\mathrm{d}z$ on $\mathbb{C} \subset \mathbb{C}_\infty$ does not extend to all of $\mathbb{C}_\infty$. Another problem is that the local expression might not transform uniquely to other points of X; for example, the meromorphic 1-form $\sqrt{z}\,\mathrm{d}z$ defined on the complex plane with the negative real axis removes (where we choose $\sqrt{1} = 1$) does not extend *uniquely* to the negative real axis. So when we use a single formula for one chart to define a meromorphic 1-form ω, the burden falls to the reader to check that the formula transforms uniquely to all of X.

[4.1] This is the content of the *Identity Theorem*, which states that if f and g are two meromorphic functions defined on a connected open set W of a Riemann surface X. Suppose that $f = g$ on a subset $S \subset W$ which has a limit point in W, then $f = g$ on W. In particular, if W is taken to be all of X, any two meromorphic functions on X that agree on an open subset of X are the same.

We want the notion of the order, previously defined for meromorphic functions, to be defined for meromorphic 1-forms as well.

Definition 4.1.7 (Order of Meromorphic 1-Forms)**.** Let ω be a meromorphic 1-form defined in a neighborhood of point p. Under some local coordinate z centered at p, we may write $\omega = f(z)\,\mathrm{d}z$ where f is a meromorphic function at $z = 0$. Then the *order of* ω *at* p, denoted $\mathrm{ord}_p(\omega)$, is defined to be $\mathrm{ord}_0(f)$, the order of the function f at 0.

One can easily check that $\mathrm{ord}_p(\omega)$ is independent of the choice of the local coordinate. Familiar properties of meromorphic functions also transform easily to meromorphic 1-forms (which are, after all, "formal expressions" involving meromorphic functions).

4.2. $\mathcal{C}^\infty$ **1-Forms.** We can relax the holomorphic or meromorphic conditions for 1-forms and obtain a notion of $\mathcal{C}^\infty$ 1-forms, which are the most general objects we wish to contour integrate. We could define $\mathcal{C}^\infty$ 1-Forms similarly as the expressions $f(x,y)\,\mathrm{d}x + g(x,y)\,\mathrm{d}y$, where $z = x + iy$ is the local variable and f, g are of class $\mathcal{C}^\infty$ with respect to x and y.

However, it is often more profitable to abandon the real and imaginary parts x and y for z and its complex conjugate $\overline{z}$. Namely,

$$x = \frac{z+\overline{z}}{2} \qquad \text{and} \qquad y = \frac{z-\overline{z}}{2i}$$

and (manually defiend)

$$\mathrm{d}x = \frac{\mathrm{d}z+\mathrm{d}\overline{z}}{2} \qquad \text{and} \qquad \mathrm{d}y = \frac{\mathrm{d}z-\mathrm{d}\overline{z}}{2i}.$$

Thus the form $f(x,y)\,\mathrm{d}x+g(x,y)\,\mathrm{d}y$ can be written instead as $r(z,\overline{z})\,\mathrm{d}z+s(z,\overline{z})\,\mathrm{d}\overline{z}$. We note that it's possible to go the other direction as well.

Definition 4.2.1 ($\mathcal{C}^\infty$ 1-Forms)**.** A $\mathcal{C}^\infty$ *1-form* on an open set $V \subset \mathbb{C}$ is an expression ω of the form

$$\omega = f(z,\overline{z})\,\mathrm{d}z + g(z,\overline{z})\,\mathrm{d}\overline{z},$$

where f and g are $\mathcal{C}^\infty$ functions on V. We say that ω is a $\mathcal{C}^\infty$ 1-form *in the coordinate* z*.*

The transformation rule is what makes sense:

Definition 4.2.2 (Transformation of $\mathcal{C}^\infty$ 1-Forms)**.** Suppose that $\omega_1 = f_1(z,\overline{z})\,\mathrm{d}z + g_1(z,\overline{z})\,\mathrm{d}\overline{z}$ and $\omega_2 = f_2(w,\overline{w})\,\mathrm{d}w + g_2(w,\overline{w})\,\mathrm{d}\overline{w}$ are $\mathcal{C}^\infty$ 1-forms defined on open sets V_1, V_2 of $\mathbb{C}$, respectively, and let $z = T(w)$ be the holomorphic transition map from V_2 to V_1. We say that ω_1 *transforms to* ω_2 *under* T if

$$f_2(w,\overline{w}) = f_1(T(w),\overline{T(w)})T'(w)$$

and

$$g_2(w,\overline{w}) = g_1(T(w),\overline{T(w)})\overline{T'(w)}.$$

Then one can define $\mathcal{C}^\infty$ 1-forms on a Riemann surface as in Definition 4.1.3.

We will need the notations of partial derivatives for defining *differentials* of functions and 1-forms. Given a $\mathcal{C}^\infty$ function $f(x,y)$, by the chain rule we have

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}$$

and
$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}.$$

Lemma 4.2.3. *A $\mathcal{C}^\infty$ function f is holomorphic on its domain iff*
$$\frac{\partial f}{\partial \overline{z}} = 0$$

Proof. Recall from Theorem 1.1.3 that if $f(x+iy) = u(x,y) + iv(x,y)$, then f is holomorphic iff it satisfies the Cauchy-Riemann equations
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This is equivalent to
$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} \\
&= \frac{1}{2}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right) - \frac{1}{2i}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) - \frac{1}{2i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \frac{1}{2i}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
&= 0.
\end{aligned}$$

□

4.3. **Differential 2-Forms and Differentials of Functions and 1-Forms.** $\mathcal{C}^\infty$ 1-forms are objects for contour integrals, but we also need the ability to do *surface integrals* on Riemann surfaces for the Stoke's Theorem. The objects that we'll integrate, in this case, are $\mathcal{C}^\infty$ 2-forms. In surface integrals on $\mathbb{C}$, we use differentials like $dx\,dy$. We can translate these into $\mathrm{d}z$ and $\mathrm{d}\overline{z}$, but we cannot multiply these formal expressions. The solution is to introduce a "formal product" - the wedge product.

Definition 4.3.1 ($\mathcal{C}^\infty$ 2-Forms on $\mathbb{C}$)**.** A $\mathcal{C}^\infty$ *2-form* on an open set $V \subset \mathbb{C}$ is an expression η of the form

$$\eta = f(z, \overline{z})\, \mathrm{d}z \wedge \mathrm{d}\overline{z}$$

where f is a $\mathcal{C}^\infty$ function on V. We say that η is a $\mathcal{C}^\infty$ 2-form *in the coordinate z.*

We mandate that wedge product formally behaves as follows:

- $\mathrm{d}z \wedge \mathrm{d}\overline{z} = -\mathrm{d}\overline{z} \wedge \mathrm{d}z$, since changing the order in the wedge product should correspond to reversing the orientation of the surface during integration, thus changing the sign of the integral, and
- $\mathrm{d}z \wedge \mathrm{d}z = \mathrm{d}\overline{z} \wedge \mathrm{d}\overline{z} = 0$, since one cannot have a surface integral using only one variable.

The transformation rule is, again, what makes sense:

Definition 4.3.2 (Transformation of $\mathcal{C}^\infty$ 2-forms)**.** Suppose that $\eta_1 = f(z, \overline{z})\, \mathrm{d}z \wedge \mathrm{d}\overline{z}$ and $\eta_2 = g(w, \overline{w})\, \mathrm{d}w \wedge \mathrm{d}\overline{w}$ are $\mathcal{C}^\infty$ 2-forms defined on open sets V_1, V_2 of $\mathbb{C}$, respectively, and let $z = T(w)$ be the holomorphic transition map from V_2 to V_1. We say that η_1 *transforms to η_2 under T* if

$$g(w, \overline{w}) = f(T(w), \overline{T(w)})|T'(w)|^2$$

(note that $|T'(w)|^2 = T'(w)\overline{T'(w)}$).

Then one can define $\mathcal{C}^\infty$ 2-forms on a Riemann surface as in Definition 4.1.3.

Now, we also need differentials of $\mathcal{C}^\infty$ functions and 1-forms. Their definitions are mainly formality and defining what makes sense. It is routine to check that they're well-defined, and we'll skip the details.

Definition 4.3.3 (Differential of $\mathcal{C}^\infty$ Functions)**.** Let f be a $\mathcal{C}^\infty$ function on a Riemann surface X, we define a a $\mathcal{C}^\infty$ 1-form $\mathrm{d}f$. For any chart $\phi : U \to V$ giving a local coordinate z, we define

$$\mathrm{d}f = \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}$$

on V. The collection of $\mathrm{d}f$ for every chart of X is a $\mathcal{C}^\infty$ 1-form on X.

Definition 4.3.4 (Differential of $\mathcal{C}^\infty$ 1-Forms)**.** Let ω be a $\mathcal{C}^\infty$ 1-form on a Riemann surface X, we define a $\mathcal{C}^\infty$ 2-form $\mathrm{d}\omega$. For any chart $\phi : U \to V$ giving a local coordinate z, we write $\omega = f(z, \overline{z})\,\mathrm{d}z + g(z, \overline{z})\,\mathrm{d}\overline{z}$ on V, then we define

$$\mathrm{d}\omega = \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \overline{z}}\right) \mathrm{d}z \wedge \mathrm{d}\overline{z}$$

on V. The collection of $\mathrm{d}\omega$ for every chart of X is a $\mathcal{C}^\infty$ 2-form on X.

There is a useful remark to make:

Lemma 4.3.5. *If ω is a holomorphic 1-form, then $d\omega = 0$.*

Proof. Since ω is a holomorphic 1-form, under a local coordinate z it has the form $\omega = f(z)\,\mathrm{d}z$, so $g = 0$. Thus

$$d\omega = \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \overline{z}}\right) \mathrm{d}z \wedge \mathrm{d}\overline{z} = 0,$$

where $\partial g/\partial z = 0$ since $g = 0$, and $\partial f/\partial \overline{z} = 0$ by Lemma 4.2.3. □

4.4. **Integration Along Paths.** We have stated that $\mathcal{C}^\infty$ 1-forms are the general objects that we will contour integrate. We thus also need a notion of paths. Recall that a *path* on a topological space X is a continuous function $[0,1] \to X$. In the case of Riemann surfaces, we want the path to possess just a bit more structure - that it is piecewise smooth. We also relax the domain to be any close interval of $\mathbb{R}$: in algebraic topology we were mainly concerned with homotopy classes of closed paths so we could reparametrize all paths to be defined on $[0,1]$, while here we wish to be more flexible.

Definition 4.4.1 (Paths on Riemann Surfaces)**.** A *path* on a Riemann surface X is a continuous and piecewise $\mathcal{C}^\infty$ function $\gamma : [a,b] \to X$. The point $\gamma(a)$ is called the *initial point* of γ and $\gamma(b)$ is called the *terminal point* of γ. We say that γ is *closed* if $\gamma(a) = \gamma(b)$.

Examples 4.4.2.

(1) Let $\gamma : [a,b] \to X$ be a path on X and $\alpha : [c,d] \to [a,b]$ is a continuous function with $\alpha(c) = a$ and $\alpha(d) = b$, then $\gamma \circ \alpha : [c,d] \to X$ is a path on X with same initial and terminal points. This is called a *reparametrization* of the path γ. Any path may be reparametrized so that its domain is $[0,1]$.

(2) Let $\gamma : [a,b] \to X$ be a path on X, then the path sending $t \in [a,b]$ to $\gamma(a+b-t)$ is called the *reversal* of γ and is denoted $-\gamma$. Its initial point is γ's terminal point, and its terminal point is γ's initial point.

(3) If $F : X \to Y$ is a $\mathcal{C}^\infty$ map (in particular if it is a holomorphic map), then $F \circ \gamma$ is a path on Y. The path $F \circ \gamma$ is often denoted $F_*\gamma$.

(4) Let γ_1 and γ_2 be two paths on X with the terminal point of γ_1 coinciding with the initial point of γ_2, then there is a path γ on X with domain $[0,1]$ such that $\gamma\big|_{[0,1/2]}$ and $\gamma\big|_{[1/2,1]}$ are reparametrizations of γ_1 and γ_2 respectively. This process is called the *concatenation* of the two paths and it can be generalized to any finite number of paths.

(5) If γ is a path on X with domain $[a,b]$, then any partition $a = a_0 < a_1 < \cdots < a_n = b$ of the interval gives a decomposition of γ into n paths, and γ is their concatenation. This is called a *partitioning* of the path γ.

(6) Let $p \in X$ be a point, and let S be a subset of X whose closure does not contain the given point p. Then there is a closed path γ on X with the following properties:
   - γ is injective and its image lies completely in the domain U of a chart ϕ.

- The closed path $\phi \circ \gamma$ on $\mathbb{C}$ has winding number 1 about the point $\phi(p)$[4.2].
- For any point $s \in S \cap U$, the winding number of $\phi \circ \gamma$ about $\phi(s)$ is 0.

We say that such a path γ is a *small path enclosing p and not enclosing any point of S.* The closed curve image(γ) divides X into two connected components, and we call the one containing p the *interior of γ.*

To define the integral of a $\mathcal{C}^\infty$ 1-form along a path, we also need the following:

Lemma 4.4.3. *Let $\gamma : [a, b] \to X$ be a path on a Riemann surface X. Then γ may be partitioned into a finite number of paths $\{\gamma_i\}$, such that each γ_i is $\mathcal{C}^\infty$ with image contained in a single chart domain of X.*

Proof. It suffices to prove for the case where γ is $\mathcal{C}^\infty$ on all of $[a, b]$ instead of piecewise $\mathcal{C}^\infty$, and the result follows from the compactness of the closed interval. For any $c \in [a, b]$, $\gamma(c)$ is in the domain of some chart $\phi_c : U_c \to V_c$ of X, so take some neighborhood $U'_c \subset U_c$ containing $\gamma(c)$, then the preimage $\phi_c^{-1}(U'_c)$ is a open subset of $[a, b]$. The collection

$$\{\phi_c^{-1}(U'_c)\}_{c\in[a,b]}$$

forms an open cover of $[a, b]$, so there is a finite subcover

$$\{\phi_{c_i}^{-1}(U'_{c_i})\}.$$

Then we partition $[a, b]$ into subintervals such that every subinterval is completely in some $\phi_{c_i}^{-1}(U'_{c_i})$, showing that images of corresponding partitioned paths is completely in the domain of ϕ_c. ☐

Now let ω be a $\mathcal{C}^\infty$ 1-form on a Riemann surface X and γ be a path on X. We can partition γ into paths $\{\gamma_i\}$ such that each γ_i is $\mathcal{C}^\infty$ on its domain $[a_{i-1}, a_i]$ and has image contained in the domain U_i of a chart ϕ_i. With respect to each chart ϕ_i, we write ω as $\omega = f_i(z, \overline{z})\,\mathrm{d}z + g_i(z, \overline{z})\,\mathrm{d}\overline{z}$, where the local variable z may be written $z = (\phi_i \circ \gamma_i)(t) := z_i(t)$ for $t \in [a_{i-1}, a_i]$.

Definition 4.4.4 (Integral of $\mathcal{C}^\infty$ 1-Forms Along Paths)**.** With the above notations, we define the *integral of ω along γ* to be the complex number

$$\int_\gamma \omega = \sum_i \int_{a_{i-1}}^{a_i} [f_i(z_i(t), \overline{z_i(t)})z'(t) + g_i(z_i(t), \overline{z_i(t)})\overline{z_i'(t)}]\,dt.$$

[4.2] Intuitively, this says that $\phi \circ \gamma$ goes around $\phi(p)$ exactly once counterclockwise in the complex plane.

Note that this makes sense since for each i where the image of γ_i is contained in the domain of a single chart $\phi_i : U_i \to V_i$, if $\omega = f\,\mathrm{d}z + g\,\mathrm{d}\overline{z}$ in this chart, then

$$\int_{\gamma_i} \omega = \int_{\phi_i\gamma_i} f\,\mathrm{d}z + g\,\mathrm{d}\overline{z}$$

is the familiar contour integral of the path $\phi_i\gamma_i$ in $V_i \subset \mathbb{C}$.

We need to check that the above definition is independent of the choice of the coordinate charts. It follows from our definition of the transformation of $\mathcal{C}^\infty$ 1-forms in Definition 4.2.2 and the fact that the integral is an invariant under refinement of partitions. Since any two partitions of γ have a common refinement, the integral is indeed well defined, depending only on the path γ and the $\mathcal{C}^\infty$ 1-form ω.

We make the following immediate remarks:

Lemma 4.4.5.

(1) *The integral is independent of the choice of parametrization. In other words,*
$$\int_{\gamma\alpha} \omega = \int_\gamma \omega$$
for any reparametrization α of the domain of γ.

(2) *The integral is linear under partition of the path, i.e., if γ is partitioned into paths $\{\gamma_i\}$, then*
$$\int_\gamma \omega = \sum_i \int_{\gamma_i} \omega.$$

(3) *The integral is $\mathbb{C}$-linear in ω:*
$$\int_\gamma (\lambda\omega_1 + \mu\omega_2) = \lambda \int_\gamma \omega_1 + \mu \int_\gamma \omega_2$$
for all $\lambda, \mu \in \mathbb{C}$.

(4) *The Fundamental Theorem of Calculus holds: if f is a $\mathcal{C}^\infty$ function defined in a neighborhood of the image of the path γ, then*
$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

(5) *If one reverse the direction of a path, the sign of the integral changes:*
$$\int_{-\gamma} \omega = -\int_\gamma \omega.$$

4.5. **Surface Integrals and Stoke's Theorem.** Mimicking Definition 4.4.4 where we've defined contour integrals on Riemann surface, to define surface integrals through a triangulable closed set $D \subseteq X$, we subdivide D into triangles, each of which lies inside the domain of a single chart. We define the integration on one triangle in one chart as follows:

Definition 4.5.1 (Integral of 2-Form through a Triangle)**.** Let T be a triangle on the Riemann surface X contained in the domain of a chart $\phi : U \to V$. Let η be a $\mathcal{C}^\infty$ 2-form on X with the form $\eta = f(z, \overline{z})\, \mathrm{d}z \wedge \mathrm{d}\overline{z}$ in this chart. Then we define

$$\begin{aligned}\iint_T \eta &= \iint_{\phi(T)} f(z, \overline{z})\, \mathrm{d}z \wedge \mathrm{d}\overline{z} \\ &= \iint_{\phi(T)} (-2i) f(x+iy, x-iy)\, dx\, dy,\end{aligned}$$

which is the usual surface integral in $\mathbb{C}$.

One may check that, when this definition is used to define surface integrals on the triangulable closed set D as mentioned, the integral $\iint_D \eta$ is well defined.

We need one more ingredient for the Stoke's Theorem.

Definition 4.5.2 (Boundary Chains)**.** Let T be a triangle on X. We construct the closed path ∂T by traversing the boundary of T counterclockwise, with the initial point and parametrization arbitrary.

Now let D by any triangulable closed set on X, we may decompose D into triangles $\{T_i\}$ and set $\partial D = \sum_i \partial T_i$, which is a *chain*[4.3], called the *boundary chain of* D.

This definition depends on the triangulation of D, but only up to partitions and reparametrizations. By Lemma 4.4.5, these won't matter when we integrate over ∂D. Thus the path integral $\int_{\partial D} \omega$ is well defined whenever ω is a $\mathcal{C}^\infty$ 1-form on X. Now we are ready to present the Stoke's Theorem on Riemann surfaces:

Stoke's Theorem. *Let D be a triangulable closed set on a Riemann surface X, and let ω be a $\mathcal{C}^\infty$ 1-form on X. Then*

$$\int_{\partial D} \omega = \iint_D d\omega.$$

[4.3]A *chain* on a Riemann surface X is a finite formal sum of paths, with integer coefficients, on which integration of $\mathcal{C}^\infty$ 1-forms is defined simply by extending path integrals by linearity.

Proof. Since both sides are additive with respect to the triangles composing a triangulation of D (by definitions of path and surface integrals as well as the boundary chain), it suffices to check for the case where D is a triangle contained inside the domain of a chart. At this point we may use the chart to transfer both integrals to the complex plane, where the theorem is simply Green's Theorem[4.4] in the complex plane. □

[4.4]The Green's Theorem states that, let C be a piecewise smooth, simple closed path in a plane oriented counterclockwise, and let D be the region bounded by C. If L and M are $\mathcal{C}^\infty$ functions of x, y defined on an open subset containing D, then

$$\int_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy.$$

One can translates this into the language of $\mathrm{d}z$ and $\mathrm{d}\overline{z}$'s.

4.6. **The Residue Theorem.** The Residue Theorem[4.5] in $\mathbb{C}$ relates a contour integral around a point with the residue at that point. We define residues on Riemann surfaces with Laurent series, which we will first define on 1-forms easily: let ω be a 1-form on a Riemann surface X, which is meromorphic at a point $p \in X$. Under a local coordinate z centered at p, ω has the Laurent series

$$\omega = f(z)\,\mathrm{d}z = \left(\sum_{n=-\infty}^{\infty} c_n z^n\right) \mathrm{d}z.$$

Definition 4.6.1 (Residue)**.** The *residue of* ω *at* p, denoted by $\mathrm{Res}_p(\omega)$, is the coefficient c_{-1} in a Laurent series for ω at p.

$\mathrm{Res}_p(\omega)$ is well defined by the following lemma, which is simply the Residue Theorem (applied to one point) in $\mathbb{C}$ when everything is transformed to $\mathbb{C}$ by a complex chart:

Lemma 4.6.2. *Let ω be a meromorphic 1-form defined in a neighborhood of $p \in X$. Let ω be a small path on X enclosing p and not enclosing any other pole of ω[4.6]. Then*

$$\mathrm{Res}_p(\omega) = \frac{1}{2\pi i}\int_\gamma \omega.$$

With some manipulation of series, we can relate the residue to the order.

Lemma 4.6.3. *Suppose f is a meromorphic at $p \in X$, then df/f is a meromorphic 1-form at p, and*

$$\mathrm{Res}_p(df/f) = \mathrm{ord}_p(f).$$

Proof. This follows from a straightforward examination of the respective Laurent series. Choose a local coordinate z centered at p, and assume that $\mathrm{ord}_p(f) = n$. Then $f = cz^n +$(higher order terms) near p with $c \neq 0$. Thus $1/f = c^{-1}z^{-n} +$(higher order terms) near p and $df = (ncz^{n-1} +$(higher order terms)$)\,\mathrm{d}z$ near p. Thus $df/f = (n/z +$(higher order terms)$)\,\mathrm{d}z$, and $\mathrm{Res}_p(df/f) = n = \mathrm{ord}_p(f)$. □

[4.5] In one-variable complex analysis, the *Residue Theorem* states that, given an open subset $U \subset \mathbb{C}$ containing a finite number of points $a_1, \cdots, a_n$, a function f holomorphic on $U \setminus \{a_1, \cdots, a_n\}$, and a closed path curve γ in U which does not pass through any of the a_k's and has winding number 1 around all a_k's in its interior, then the contour integral

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{a_k \text{ inside } \gamma} \mathrm{Res}(f, a_k).$$

[4.6] See (6) of Examples 4.4.2.

The Residue Theorem in $\mathbb{C}$ states that the sum of residues at some points is equal to some integral. We also have the Residue Theorem for compact Riemann surfaces, which is even simpler:

The Residue Theorem. *Let ω be a meromorphic 1-form on a compact Riemann surface X. Then*

$$\sum_p \operatorname{Res}_p(\omega) = 0.$$

Proof. Firstly, note that this is indeed a finite sum since $\operatorname{Res}_p(\omega) = 0$ unless p is a pole of ω. Let $p_1, p_2, \cdots, p_n$ be the poles of ω. For each p_i, choose a small path γ_i enclosing p_i and no other pole of ω, and let U_i be the interior of γ_i[4.7]. By Lemma 4.6.2, we have

$$\int_{\gamma_i} \omega = 2\pi i \operatorname{Res}_{p_i}(\omega)$$

for each i. Let $D = X \setminus \bigcup_i U_i$, then D is triangulable, and $\partial D = -\sum_i \gamma_i$ is a chain on X[4.8]. Therefore,

$$\begin{aligned}
\sum_i \operatorname{Res}_{p_i}(\omega) &= \frac{1}{2\pi i} \sum_i \int_{\gamma_i} \omega \\
&= -\frac{1}{2\pi i} \int_{-\sum_i \gamma_i} \omega \\
&= -\frac{1}{2\pi i} \int_{\partial D} \omega \\
&= -\frac{1}{2\pi i} \iint_D \mathrm{d}\omega \\
&= 0,
\end{aligned}$$

where the last equality follows from Lemma 4.3.5 because ω is holomorphic in a neighborhood of D. □

Applying the Residue Theorem to df/f, and using Lemma 4.6.3, we have proven the following corollary again:

Corollary 4.6.4. *Let f be a nonconstant meromorphic function on a compact Riemann surface X. Then*

$$\sum_{p \in X} \operatorname{ord}_p(f) = 0.$$

[4.7] See (6) of Examples 4.4.2 for the interior of an enclosing path.

[4.8] See (2) of Examples 4.4.2 for $-\gamma$, the reversal of γ. This notion is extended to chains under linearity.

5. Divisors

5.1. **Definition and Examples of Divisors.** Divisors are some sort of formal objects that packages the different points on a Riemann surface as well as values associated to them (for example, the zeroes and poles of a meromorphic function as well as their degrees, or the branching points of a holomorphic map as well as their multiplicities).

Definition 5.1.1 (Divisors)**.** A *divisor* on a Riemann surface X is a function $D : X \to \mathbb{Z}$ that is whose support[5.1] is a discrete subset of X. The divisors on X form a group (a subgroup of $\mathbb{Z}^X$, the group of all functions $X \to \mathbb{Z}$, indeed) under pointwise addition, denoted by $\mathrm{Div}(X)$.

However, we will usually denote a divisor D using a summation notation, and write

$$D = \sum_{p \in X} D(p) \cdot p,$$

where the set of p such that $D(p) \neq 0$ is discrete. Still keep in mind that divisors are defined as functions, though.

Definition 5.1.2 (Degree of Divisors)**.** The *degree* of a divisor D on a compact Riemann surface is the finite sum

$$\deg(D) = \sum_{p \in X} D(p).$$

Let X be a Riemann surface. We will see a lot of examples of divisors, all of which correspond to concepts that we've seen:

Examples 5.1.3.

(1) Let f be a nonzero meromorphic function on X. The *divisor of f*, denoted by $\mathrm{div}(f)$, is the divisor defined by the order function:

$$\mathrm{div}(f) = \sum_{p} \mathrm{ord}_p(f) \cdot p.$$

Any divisor of this form is called a *principal divisor* on X. The set of principal divisors form a subgroup of $\mathrm{Div}(X)$, denoted by $\mathrm{PDiv}(X)$.

The result that we've seen twice (Proposition 3.4.5 and Corollary 4.6.4) translates to

Lemma 5.1.4. *If f is a nonzero meromorphic function on a compact Riemann surface, then* $\deg(\mathrm{div}(f)) = 0$.

[5.1] The *support* of D is the set of points $p \in X$ where $D(p) \neq 0$.

(2) Again let f be a nonzero meromorphic function on X, the *divisor of zeroes of* f, denoted by $\mathrm{div}_0(f)$ is the divisor

$$\mathrm{div}_0(f) = \sum_{p \text{ zero of } f} \mathrm{ord}_p(f) \cdot p.$$

Similarly, the *divisor of poles of* f, denoted by $\mathrm{div}_\infty(f)$, is the divisor

$$\mathrm{div}_\infty(f) = \sum_{p \text{ pole of } f} (-\mathrm{ord}_p(f)) \cdot p.$$

Both of these divisors are nonnegative functions with disjoint support, such that

$$\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f).$$

(3) Let ω be a nonzero meromorphic 1-form on X. The *divisor of* ω, denoted by $\mathrm{div}(\omega)$, is the divisor defined by the order function:

$$\mathrm{div}(\omega) = \sum_p \mathrm{ord}_p(\omega) \cdot p.$$

Any divisor of this form is called a *canonical divisor* on X. The set of canonical divisors on X is denoted by $\mathrm{KDiv}(X)$.

Since the ratio of two meromorphic functions is meromorphic, it's easy to see that given two meromorphic 1-forms ω_1 and ω_2 on X, with ω_1 not identically zero, we can find a unique meromorphic function f on X with $\omega_2 = f\omega_1$. This exactly translate to:

Corollary 5.1.5. *The set* $\mathrm{KDiv}(X)$ *is exactly a coset of the subgroup* $\mathrm{PDiv}(X)$*. In other words, the difference of any two canonical divisors is principal.*

In fact, the Riemann-Hurwitz Formula allows us to compute the degree of a canonical divisor on a compact Riemann surface as well:

Proposition 5.1.6. *If X is a compact Riemann surface with genus g, then any canonical divisor on X has degree* $2g-2$.

Proof. From the previous corollary, we see that the degree of any two canonical divisors on X have the same degree, thus we just want to select an easy one to compute. However, note that we know nothing about X while we have a complete knowledge of meromorphic functions on $\mathbb{C}_\infty$, we want a method to transform between meromorphic functions (and 1-forms) between X and $\mathbb{C}_\infty$.

Way back in Remark 3.1.3, we have seen that given a nonconstant holomorphic map $F : X \to Y$ between Riemann surfaces, any meromorphic function f on Y may be "pulled back" to a nonconstant meromorphic function $f \circ F$ on X. We will use a similar idea here for meromorphic 1-forms. The *pullback* of a meromorphic 1-form ω via F, denoted $F^*\omega$), is defined by

$$F^*\omega = f(h(w), \overline{h(w)})h'(w)\,\mathrm{d}w + g(h(w), \overline{h(w)})\overline{h'(w)}\mathrm{d}\overline{w},$$

where $\omega = f(z, \overline{z})\,\mathrm{d}z + g(z, \overline{z})\,\mathrm{d}\overline{z}$ in local variables z in Y. The key is that we can track associated series and conclude that

$$\mathrm{ord}_p(F^*\omega) = (1 + \mathrm{ord}_{F(p)}(\omega))\,\mathrm{mult}_p(F) - 1,$$

the proof of which is very similar to that of Lemma 4.6.3.

Now consider the easiest meromorphic 1-form ω on $\mathbb{C}_\infty$ that we've defined in Example 4.1.5, with $\omega = \mathrm{d}z$ near 0 and $\omega = -1/w^2\,\mathrm{d}w$ near ∞. We have $\mathrm{ord}_\infty(\omega) = -2$ and $\mathrm{ord}_p(\omega) = 0$ anywhere else, so $\deg(\mathrm{div}(\omega)) = -2$. We want to pull this meromorphic 1-form back to X.

Although highly nontrivial, we are going to assume that *there exists a nonconstant meromorphic function f on X*. Therefore we get the associated nonconstant holomorphic map $F : X \to \mathbb{C}_\infty$, via which we pull back ω to X. Let $\eta = F^*\omega$, a meromorphic 1-form on X, whose divisor's degree we now calculate:

$$\begin{aligned}
\deg(\mathrm{div}(\eta)) &= \sum_{p\in X} \mathrm{ord}_p(\eta) \\
&= \sum_{p\in X} \mathrm{ord}_p(F^*\omega)) \\
&= \sum_{p\in X} [(1 + \mathrm{ord}_{F(p)}(\omega))\,\mathrm{mult}_p(F) - 1] \\
&= \sum_{\substack{q\neq\infty \\ p\in F^{-1}(q)}} [\mathrm{mult}_p(F) - 1] + \sum_{p\in F^{-1}(\infty)} [-\,\mathrm{mult}_p(F) - 1] \\
&= \sum_{p\in X} [\mathrm{mult}_p(F) - 1] - 2 \sum_{p\in F^{-1}(\infty)} \mathrm{mult}_p(F) \\
&= 2g - 2 + 2\deg(F) - 2\deg(F) \\
&= 2g - 2,
\end{aligned}$$

□

where the second last equality follows from the Riemann-Hurwitz Formula.

(4) Let $F : X \to Y$ be a nonconstant holomorphic map, and let q be a point of Y. The *inverse image divisor of* q, denoted by $F^*(q)$, is the divisor on X defined by

$$F^*(q) = \sum_{p \in F^{-1}(q)} \operatorname{mult}_p(F) \cdot p.$$

Moreover, we may extend this definition to any divisor D on Y. The pullback of the divisor $D = \sum_{q \in Y} D(q) \cdot q$ is simply

$$F^*(D) = \sum_{q \in Y} D(q) F^*(q) = \sum_{q \in Y} D(q) \left[\sum_{p \in F^{-1}(q)} \operatorname{mult}_p(F) \cdot p \right].$$

In function form, we have

$$F^*(D)(p) = \operatorname{mult}_p(F) D(F(p))$$

for all $p \in X$.

(5) Let $X \to Y$ be a nonconstant holomorphic map. The *ramification divisor* of F, denoted by R_F, is the divisor on X defined by

$$R_F = \sum_{p \in X} [\operatorname{mult}_p(F) - 1] \cdot p.$$

The *branch divisor* of F, denoted by B_F, is the divisor on Y defined by

$$B_F = \sum_{y \in Y} \left[\sum_{p \in F^{-1}(y)} (\operatorname{mult}_p(F) - 1) \right] \cdot y.$$

If X and Y are compact, then both sums are finite, as we've seen. It's immediate that $\deg(R_F) = \deg(B_F)$. The Riemann-Hurwitz Formula translates to

$$\chi(X) = \deg(F)\chi(Y) - \deg(R_F).$$

5.2. **Structures on Divisors: the Partial Ordering, the Spaces $L(D)$ and $L^{(1)}(D)$.** Having seen a lot of examples of divisors, we will start to place more structures on them. There is an intuitive way to place a partial ordering on $\mathrm{Div}(X)$, if we think about divisors as functions. We write $D \geq 0$ if $D(p) \geq 0$ for all p, $D > 0$ if $D \geq 0$ and $D \neq 0$, $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$, and similarly for $>$. The relations $<$ and $\geq$ are also defined similarly. Note that every divisor D can be uniquely written in the form

$$D = P - N$$

where P and N are nonnegative divisors with disjoint support; an important example is

$$\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f)$$

when f is a meromorphic function.

With this language we will define a very important notion which is used to organize meromorphic functions on a Riemann surface:

Definition 5.2.1 (The Space $L(D)$)**.** The *space of meromorphic functions with poles bounded by D,* denoted by $L(D)$, is the set of meromorphic functions

$$L(D) = \{f \in \mathcal{M}(X) : \mathrm{div}(f) \geq -D\}.$$

To deal with the zero function, we assume that $\mathrm{ord}_p(f) = \infty$ if f is constantly zero on a neighborhood of p.

We note that $L(D)$ is a complex vector space. The terminology of $L(D)$ is used because if $D(p) = n > 0$ then f cannot have a pole at p with order more than n, and if $D(p) = -n < 0$ then $D(p)$ must have a zero at p with order more than n.

It is obvious that if $D_1 \leq D_2$, then D_2 poses a stricter condition on the poles and zeros of f, thus

$$L(D_1) \subseteq L(D_2) \text{ if } D_1 \leq D_2.$$

A meromorphic function is holomorphic if it has no poles, i.e. $\mathrm{div}(f) \geq 0$, thus

$$L(0) = \mathcal{O}(X).$$

In particular, if X is compact, then

$$L(0) = \{\text{constant functions on } X\} \cong \mathbb{C},$$

by Corollary 2.3.4.

We also have the following quick result:

Lemma 5.2.2. *Let X be a compact Riemann surface. If D is a divisor on X with $\deg(D) < 0$, then $L(D) = 0$.*

Proof. Assume for contradiction that $f \in L(D)$ and f is not identically zero, then the divisor $E = \mathrm{div}(f) + D \geq 0$ but $\deg(E) = \deg(\mathrm{div}(f)) + \deg(D) = \deg(D) < 0$, which is a contradiction. □

We also have a very similar construction for meromorphic 1-forms:

Definition 5.2.3 (The Space $L^{(1)}(D)$)**.** The *space of meromorphic 1-forms with poles bounded by* D, denoted by $L^{(1)}(D)$, is the set of meromorphic 1-forms

$$L^{(1)}(D) = \{\omega \in \mathcal{M}^{(1)}(X) : \mathrm{div}(\omega) \geq -D\}.$$

We note that when $D = 0$, the space $L^{(1)}(0)$ contains all meromorphic 1-forms with nonnegative orders everywhere, which is exactly the space of holomorphic 1-forms, $\Omega^1(X)$.

Since the div function inherits the property $\mathrm{div}(f\omega) = \mathrm{div}(f) + \mathrm{div}(\omega)$ from the order function on meromorphic functions and 1-forms, we have an isomorphism:

Proposition 5.2.4. *Fix a canonical divisor* $K = \mathrm{div}(\omega)$ *and another divisor* D*. Then the multiplication map*

$$\mu_\omega : L(D + K) \to L^{(1)}(D)$$

defined by $f \mapsto f\omega$ *is an isomorphism of vector spaces. In particular,* $\dim L^{(1)}(D) = \dim L(D + K)$.

Proof. First we need to verify that whenever $f \in L(D + K)$, $f\omega \in L^{(1)}(D)$, but this follows from the property above. The map is clearly linear and injective. Surjectivity follows from Corollary 5.1.5: given $\omega' \in L^{(1)}(D)$, the difference

$$\mathrm{div}(\omega') - \mathrm{div}(\omega) = \mathrm{div}(\omega') - K$$

is a principal divisor, say $\mathrm{div}(f)$. Therefore,

$$\mathrm{div}(f) + D + K = \mathrm{div}(\omega') - K + D + K = \mathrm{div}(\omega') + D \geq 0,$$

so $f \in L(D + K)$. □

We are very interested in the dimension of $L(D)$, since it measures the size of the space of meromorphic functions on X with certain properties (bounded poles and zeros). The rest of this subsection would be devoted to proving that any $L(D)$ is finite-dimensional.

Lemma 5.2.5. *Let* X *be a Riemann surface and* D *a divisor on* X*. For any point* p *of* X*, whether* $L(D - p) = L(D)$ *or* $\dim L(D - p) = \dim L(D) - 1$.

Proof. Choose a local coordinate z centered at p and let $n = -D(p)$. Every function f in $L(D)$ has order at least n at p, so its Laurent series at p has the form cz^n+(higher order terms), where c is allowed to be zero. Define a map

$$\begin{aligned} \alpha : L(D) &\to \mathbb{C} \\ f &\mapsto c, \end{aligned}$$

then α is a linear map with kernel $L(D-p)$. α is either identically 0 or surjective; in the former case $L(D-p) = L(D)$, and in the latter case $\dim L(D-p) = \dim L(D) - 1$. □

Proposition 5.2.6. *Let X be a compact Riemann surface and let D be a divisor on X, then the space $L(D)$ is finite-dimensional.*

Proof. Write $D = P - N$ uniquely as a different between nonnegative divisors with disjoint support; we will show that $\dim L(D) \leq 1 + \deg(P)$. We will induct on the degree of the positive part P of D. For the base case where $\deg(P) = 0$, we have $P = 0$ and $\dim L(P) = 1$. Since $D \leq P$, we have $L(D) \subseteq L(P)$ so $\dim L(D) \leq \dim L(P) = 1 = 1 + \deg(P)$, as required.

Now suppose the positive part P of D has degree $k \geq 1$. Choose a point p in the support of P, so $P(p) \geq 1$. Consider the divisor $D - p$; its positive part is $P - p$, which has degree $k - 1$. By the induction hypothesis,

$$\dim(D-p) \leq \deg(P-p) + 1 = \deg(P),$$

and by the previous lemma, we have

$$\dim L(D) \leq 1 + \dim L(D-p) \leq \deg(P) + 1,$$

as required. □

5.3. **More Structures on Divisors: Linear Equivalence and Complete Linear Systems.** In this subsection we present the definition of linear equivalence of divisors and complete linear systems. While $L(D)$ is a space of certain meromorphic functions on X, the complete linear system is a space of certain divisors on X; as we will see, the size of the two are tightly related.

Definition 5.3.1 (Linear Equivalence)**.** Let X be a Riemann surface. Two divisors D_1 and D_2 on X are said to be *linearly equivalent*, denoted $D_1 \sim D_2$, if their difference is a principal divisor, i.e. their difference is the divisor of a meromorphic function.

It is easy to check that linear equivalence is an equivalence relation on the set $\text{Div}(X)$, and that if $D_1 \sim D_2$ then $\deg(D_1) = \deg(D_2)$.

Examples 5.3.2.

(1) Let f be a nonzero meromorphic function on a Riemann surface X, then $\text{div}_0(f) \sim \text{div}_\infty(f)$, since their difference is simply $\text{div}(f)$.
(2) Any two canonical divisors on X are linearly equivalent, since their difference is the divisor of the ratio of two meromorphic 1-forms, which is a meromorphic function.
(3) Any two points on the Riemann Sphere $\mathbb{C}_\infty$, considered as divisors on $\mathbb{C}_\infty$, are linearly equivalent. More generally, we have the following proposition for $\mathbb{C}_\infty$:

Proposition 5.3.3. *Let D_1 and D_2 be two divisors on the Riemann Sphere. Then $D_1 \sim D_2$ iff $\deg(D_1) = \deg(D_2)$.*

Proof. Let $D = D_1 - D_2$, then it suffices to prove that D is a principal divisor iff $\deg(D) = 0$. We have seen that the condition is necessary. For see that it's sufficient, suppose that $\deg(D) = 0$, so

$$D = \sum_i e_i \cdot \lambda_i + e_\infty \cdot \infty$$

where the λ_i are points on $\mathbb{C}$ and $e_\infty = -\sum_i e_i$. Then $D = \text{div}(f)$, where

$$f(z) = \prod_i (z - \lambda_i)^{e_i}$$

is a meromorphic function on $\mathbb{C}_\infty$. □

Now we present the definition of complete linear systems of divisors:

Definition 5.3.4 (Complete Linear Systems of Divisors)**.** Let X be a Riemann surface and D a divisor on X. The *complete linear system of* D, denoted $|D|$, is the set of all nonnegative divisors $E \geq 0$ which are linearly equivalent to D:

$$|D| = \{E \in \mathrm{Div}(X) : E \sim D \text{ and } E \geq 0\}.$$

As we've noted in the beginning of the subsection, the size of $|D|$ is closely related to that of $L(D)$. In fact, consider the following map:

Proposition 5.3.5. *Define a map*

$$S : \mathbb{P}(L(D)) \to |D|$$

from the projectivization[5.2] *of* $L(D)$ *to the complete linear system of* D *by sending the span of a meromorphic function* $f \in L(D)$ *to the divisor* $\mathrm{div}(f) + D$*. Then* S *is well defined and a bijection between* $\mathbb{P}(L(D))$ *and* $|D|$ *when* X *is a compact Riemann surface.*

Proof. It's easy to see that S is well defined, since $\mathrm{div}(\lambda f) = \mathrm{div}(f)$ for any nonzero constant λ. Now, we prove that S is a bijection.

Suppose that $S(f) = S(g)$ for functions $f, g \in L(D)$. Thus $\mathrm{div}(f) + D = \mathrm{div}(g) + D$ and $\mathrm{div}(f) = \mathrm{div}(g)$. Therefore $\mathrm{div}(f/g) = 0$, and since X is compact, f/g is a nonzero constant function. This shows that f and g have the same span in $L(D)$, i.e. S is injective. On the other hand, given a divisor $E \in |D|$. Since $E \sim D$, there exists a meromorphic function f such that $E = \mathrm{div}(f) + D$; moreover, $E \geq 0$, so $\mathrm{div}(f) \geq -D$ and $f \in L(D)$. Thus $S(f) = E$, so S is surjective. □

Therefore, we define $\dim|D|$ of a divisor D on a compact Riemann surface to be $\dim L(D) - 1$.

Sometimes it's useful to be able to select divisors from a linear system with support away from the prescribed points, so we will need one more piece of definition.

Definition 5.3.6 (Base Points of Linear Systems)**.** Let $|D|$ be a complete linear system on a Riemann surface X. A point p is a *base point* of $|D|$ if every divisor $E \in Q$ contains p. A complete linear system $|D|$ is said to be *base-point-free* if it has no base points.

Recall that Lemma 5.2.5 says $\dim L(D - p)$ is either $\dim L(D)$ or $\dim L(D) - 1$. This is exactly related to whether p is a base point of D or not; intuitively, if p is in every divisor of $|D|$, removing p wouldn't change the size of $L(D)$ and vice versa.

[5.2] The projectivization of a complex vector space V, denoted $\mathbb{P}(V)$, is set of all 1-dimensional subspaces of V. In particular, if $\dim V = n+1$, then $\mathbb{P}(V)$ is bijective to $\mathbb{P}^n$.

Proposition 5.3.7. *A point p is a base point of the complete linear system $|D|$ iff $L(D-p) = L(D)$. Therefore, $|D|$ is base-point-free iff $\dim L(D-p) = \dim L(D) - 1$ for all $p \in X$.*

Proof. p is a base point of $|D|$ iff $D'(p) \geq 1$ for all divisors $D' \in |D|$. Since $D' = \text{div}(f) + D$ for some meromorphic function $f \in L(D)$, this is equivalent to

$$\begin{aligned} & D'(p) = \text{ord}_p(f) + D(p) \geq 1 \\ \Leftrightarrow \quad & \text{ord}_p(f) \geq -D(p) + 1 \\ \Leftrightarrow \quad & \text{div}(f) \in L(D-p). \end{aligned}$$

Since $D - p \leq D$ as divisors, $L(D-p) \subseteq L(D)$, so p is a base point iff the other inclusion holds and $L(D-p) = L(D)$. □

Given a complete linear system $|D|$, we can also decompose it to the sum of a fixed divisor and a part that has no base points. In specific, let $F = \min\{E : E \in |D|\}$ and write $|D| = F + |D - F|$. F is a nonnegative divisor and is called the *fixed part* of $|D|$, while $|D - F|$ has no fixed points and is called the *moving part* of D.

6. Algebraic Curves and the Riemann-Roch Theorem

6.1. **Algebraic Curves.** As we've mentioned, the fact that compact Riemann surfaces have nonconstant meromorphic functions is highly nontrivial. The proof, by producing meromorphic functions for an unknown compact Riemann surface, is mostly technical and involves a lot of functional analysis. And the end of the section, the Riemann-Roch Theorem will help us compute the dimension of the vector space $L(D)$, and thus quantitatively show that there exist a *wealth* of nontrivial meromorphic functions on a compact Riemann surface.

Once we assume that every compact Riemann surface has nonconstant meromorphic functions, our theory will be mostly algebraic. Thus in the following definition we will assume our Riemann surface is equipped with reasonably handy meromorphic functions:

Definition 6.1.1 (Algebraic Curves)**.** Let S be a set of meromorphic functions on a compact Riemann surface X. We say that S *separates points* of X if for every pair of distinct points p and q in X there is a meromorphic function $f \in S$ such that $f(p) \neq f(q)$. We say that S *separates tangents* of X if for every point $p \in X$ there is a meromorphic function $f \in S$ whose associated function to $\mathbb{C}_\infty$ has multiplicity one at p. A compact Riemann surface X is an *algebraic curve* if the field $\mathcal{M}(X)$ of global meromorphic functions separates the points and tangents of X.

In fact, the basic result that we will not prove is the following, which shows that all compact Riemann surfaces are supplied with point-separating and tangent-separating meromorphic functions. We will, however, continue to distinguish the two notions and make it explicit when results only apply to algebraic curves.

Theorem 6.1.2. *Every compact Riemann surface is an algebraic curve.*

Remark 6.1.3. There is actually an equivalence between the category of compact Riemann surfaces (morphisms are nonconstant holomorphic maps) and the category of smooth irreducible projective algebraic curves over $\mathbb{C}$ (morphisms are non-constant regular maps). Thus having studied Riemann surfaces and their morphisms (non-constant holomorphic maps), we know quite a lot about algebraic curves as well, actually.

Having access to some reasonably useful meromorphic functions, we are able to show the existence of a lot more meromorphic functions with desirable functions. Our ultimate goal would be the Laurent Series Approximation Lemma, which roughly says that we can construct meromorphic functions on X that has prescribed local behavior at a finite number of points.

A Laurent polynomial $r(z) = \sum_{i=n}^{m} c_i z^i$ is called a *Laurent tail* of a Laurent series $h(z)$ if the Laurent series starts with $r(z)$, i.e. if all terms of $h-r$ are of degree higher than m. A Laurent tail thus roughly captures the essence of the behavior of a Laurent series (e,g. the order of poles and zeroes) at 0, since all higher-degree terms become irrelevant near 0.

The remainder of the section would be dedicated to proving the Laurent Series Approximation Lemma. We will do this in a series of lemmas that build off each other. Mostly it will be a rush through technicalities, and we only highlight the key construction in each lemma.

Lemma 6.1.4. *Let X be an algebraic curve, and let $p \in X$. Then for any integer N there is a global meromorphic function f on X with* $\mathrm{ord}_p(f) = N$.

Proof. First we find a meromorphic function g whose associated holomorphic map to $\mathbb{C}_\infty$ has multiplicity 1 at p, then g is either holomorphic at p or g has a simple pole at p. In the former case, $g - g(p)$ has order 1 at p, and in the latter case, $1/g$ does. In either case we have a meromorphic function h with $\mathrm{ord}_p(h) = 1$. Then

$$f = g^N$$

has order N at p. □

Lemma 6.1.5. *Let X be an algebraic curve. Fix a point p on X and a local coordinate z centered at p. Fix any Laurent polynomial $r(z)$ in z, then there exists a global meromorphic function f on X whose Laurent series at p has $r(z)$ as a Laurent tail.*

Proof. We will induct on the number of terms of $r(z)$. The case where $r(z)$ is a monomial cz^m is handled by the previous lemma. Now suppose that $r(z) = \sum_{i=n}^{m} c_i z^i$ has at least two terms. We can find a global meromorphic function h with $c_n z^n$ as a Laurent tail, and by the induction hypothesis we can also find a global meromorphic function g with the tail of $h - r$ at p (since this tail has less terms than $r(z)$. The function

$$f = h - g$$

then has r as a Laurent tail. □

Now we begin to approximate simultaneously at multiple points.

Lemma 6.1.6. *Let X be an algebraic curve. Then for any two points p and q in X, there is a global meromorphic function f on X with a zero at p and a pole at q.*

Proof. Since $\mathcal{M}(X)$ separates points of X, there is a global meromorphic function g on X such that $g(p) \neq g(q)$. By replacing g by $1/g$ if necessarily, we can make that p is not a pole of g. By replacing g by $g - g(p)$, we can make p a zero of g. If q is a pole of g, we are done; otherwise,

$$f = g/(g(q) - g)$$

has a pole at q and maintains a zero at p. □

Lemma 6.1.7. *Let X be an algebraic curve. Then for any finite number of points $p, q_1, \cdots, q_n$ in X, there is a global meromorphic function f on X with a zero at p and a pole at each q_i.*

Proof. We will induct on the number n. The $n = 1$ case is the previous lemma. Suppose that $n \geq 2$, and the induction hypothesis gives us a global meromorphic function g on X with a zero at p and a pole at $q_1, \cdots, q_{n-1}$. Let h be a global meromorphic function on X with a zero at p and a pole at q_n, then we claim that

$$f = g + h^m$$

for large m has the required zeroes and poles.

Certainly f has a zero at p. For any $i = 1, 2, \cdots, n-1$, g has a pole at q_i. If h is holomorphic at q_i, then f has a pole at q_i for every m; if h has a pole at q_i, for large m the pole of h^m at q_i would be greater than the order than the pole of g, so the sum $f = g + h^m$ has a pole at q_i. Finally, since h has a pole at q_n, f has a pole at q_n for large m for the same reasons. □

Lemma 6.1.8. *Let X be an algebraic curve. Then for any finite number of points $p, q_1, \cdots, q_n$ in X, and any $N \geq 1$, there is a global meromorphic function f on X with* $\mathrm{ord}_p(f-1) \geq N$ *and* $\mathrm{ord}_{q_i} \geq N$ *for each i.*

Proof. Let g be a global meromorphic function with a zero at p and a pole at each q_i, then

$$f = 1/(1 + g^N)$$

has the required properties. □

Lemma 6.1.9 (Laurent Series Approximation)**.** *Suppose that X is an algebraic curve. Fix a finite number of points $p_1, \cdots, p_n$ in X, choose a local coordinate z_i at each p_i, and finally choose Laurent polynomials $r_i(z_i)$ for each i. Then there exists a global meromorphic function f on X such that f has r_i as a Laurent tail at p_i for every i.*

Proof of the Laurent Series Approximation Lemma. By Lemma 6.1.5, there are global meromorphic functions g_i on X such that g_i has r_i as a Laurent tail at p_i. Now we wish to piece the g_i's together in a suitable way. We do this by taking a combination of their product with suitable functions h_i, where each h_i is very close to 1 near p_i but very close to 0 near other p_j's.

Fix an integer N larger than the degree of every r_i. For a global meromorphic function f on X to have r_i as a Laurent tail at p_i is equivalent to saying $\mathrm{ord}_{p_i}(f - r_i) \geq N$. Let M be the minimum of the orders $\mathrm{ord}_{p_i}(r_i)$, which is the same as the minimum of $\mathrm{ord}_{p_i}(g_i)$. By by previous lemma, there are global meromorphic functions h_i on X such that for each i, $\mathrm{ord}_{p_i}(h_i - 1) \geq N - M$ and $\mathrm{ord}_{p_j}(h_i) \geq N \geq M$ for all $j \neq i$.

Now consider the function

$$f = \sum_i h_i g_i.$$

At each p_i, the term $h_i g_i$ has r_i as its Laurent tail, and all other $h_j g_j$ where $j \neq i$ is zero up through order $N - 1$. Thus f has r_i as its Laurent tail at each p_i. □

Corollary 6.1.10. *Let X be an algebraic curve. Fix a finite number of points $p_1, \cdots, p_n$ in X, and a finite number of integers m_i. Then there exists a global meromorphic function f on X such that $\mathrm{ord}_{p_i}(f_i) = m_i$ for every i.*

6.2. **Laurent Tail Divisors and Finite-Dimensionality of $H^1(D)$.** The Laurent Series Approximation Lemma implies at the need for an object capturing the Laurent series tails defined on a finite set of points of a compact Riemann surface X. Divisors are exactly made for this, only that we now associate Laurent polynomials instead of integers to points:

Definition 6.2.1 (Laurent Tail Divisors)**.** Let X be a compact Riemann surface. For each point p in X, fix a local coordinate z_p centered at p, so we can associate a Laurent series in the coordinate z_p to any meromorphic function defined near p. Now, a *Laurent tail divisor* on X is a finite formal sum

$$\sum_p r_p(z_p) \cdot p,$$

where $r_p(z_p)$ is a Laurent polynomial in the coordinate z_p. The set of Laurent tail divisors on X forms a group under formal addition, and will be denoted by $\mathcal{T}(X)$.

Furthermore, given an ordinary divisor D on X, we have the subgroup

$$\mathcal{T}[D](X) = \left\{ \sum_p r_p \cdot p : \text{for all } p \text{ with } r_p \neq 0, \text{ the top term of } r_p \right.$$
$$\left. \text{has degree strictly less than } -D(p) \right\}.$$

Note that Laurent tail divisors are not divisors that we have defined, since their target spaces as functions is the space of Laurent polynomials instead of $\mathbb{Z}$.

There is natural truncation map from $\mathcal{T}(X) \to \mathcal{T}[D](X)$ that takes each nonzero Laurent polynomial r_p and removes all terms of degree $-D(p)$ and higher. Likewise, if $D_1 \leq D_2$ are divisors, then there is a truncation map

$$t^{D_1}_{D_2} : \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X)$$

defined by removing from each r_p all terms of degree $-D_2(p)$ and higher.

Given a meromorphic function f and a divisor D, we also have the multiplication map

$$\mu^D_f : \mathcal{T}[D](X) \to \mathcal{T}[D - \operatorname{div}(f)](X)$$

defined by sending each r_p to the suitable truncation of fr_p. Note that μ^D_f is an isomorphism with inverse $\mu^{D-\operatorname{div}(f)}_{1/f}$.

Finally, given a divisor D, we have a map

$$\alpha_D : \mathcal{M}(X) \to \mathcal{T}[D](X)$$

defined by sending the meromorphic function f to the Laurent tail divisor $\sum_p r_p \cdot p$, where each r_p is the suitable truncation of the Laurent series of f at p using coordinates z_p. This map apparently behaves well with truncation maps and multiplication maps. If $D_1 \leq D_2$ are divisors, then α_{D_2} is the composition $t^{D_2}_{D_1} \circ \alpha_{D_1}$:

$$\begin{tikzcd} \mathcal{M}(X) \arrow[r, "\alpha_{D_1}"] \arrow[rr, bend right, "\alpha_{D_2}"'] & T[D_1](X) \arrow[r, "t^{D_1}_{D_2}"] & T[D_2](X). \end{tikzcd}$$

And if f an g are meromorphic functions on X and D is any divisor, then

$$\mu_f(\alpha_D(g)) = \alpha_{D-\mathrm{div}(f)}(fg).$$

In language of Laurent tail divisors and this map α_D, we have some restatement of existing results. If $D(p) = 0$ for point p (which happens at all but finitely many points), then $(\alpha_D(f))(p) = 0$ iff the Laurent series of f at p contains only terms of nonnegative degrees, which is equivalent to that f is holomorphic at p. Also, the space $L(D)$ is the space of meromorphic functions on X with order at least $-D(p)$ at each p, thus

$$L(D) = \ker(\alpha_D).$$

6.3. **The Mittag-Leffler Problem and $H^1(D)$.** If we take a Laurent tail divisor $Z \in T[D](X)$, we want to ask whether it is in the image of α_D. This is equivalent to asking if there is a global meromorphic function with precisely these tails; since Laurent series tails largely captures the local behavior of functions, we are asking if the given local conditions induces a global meromorphic function. Note that this is much harder than what we have accomplished in the Laurent Series Approximation Lemma: where $D(p) = 0$ (this happens at all but finitely many points), the *absence* of a Laurent tail r_p in Z implies that f is holomorphic at p. The problem of constructing functions with specified Laurent tails at a finite number of points, and no other poles, is called the *Mittag-Leffler Problem* for the Riemann surface X.

Algebraically, the Mittag-Leffler Problem of constructing images to α_D is measured by the cokernel[6.1].

Definition 6.3.1 ($H^1(D)$)**.** Given a divisor D on a compact Riemann surface X, we have the map $\alpha_D : \mathcal{M}(X) \to \mathcal{T}[D](X)$ and define

$$H^1(D) = \operatorname{coker}(\alpha_D) = \mathcal{T}[D](X)/\operatorname{image}(\alpha_D).$$

What we mean by $H^1(D)$ "measures" the Mittag-Leffler Problem is this: by definition, a Laurent tail divisor $Z \in \mathcal{T}[D](X)$ is in the image of α_D iff its coset in $H^1(D)$ is zero.

It's crucial to compute the dimension of $H^1(D)$, since $H^1(D) = 0$ implies that all Laurent tail divisors in $\mathcal{T}[D](X)$ have preimages. Exact sequences[6.2] facilitate the investigation of the dimension of $H^1(D)$. We have the following exact sequence

$$0 \to L(D) \xrightarrow{\iota} \mathcal{M}(X) \xrightarrow{\alpha_D} \mathcal{T}[D](X) \xrightarrow{\pi} H^1(D) \to 0,$$

where ι is the inclusion of $L(D)$ in $\mathcal{M}(X)$ and π is the projection homomorphism to quotient groups. This may be written as the short exact sequence[6.3]

$$0 \to \mathcal{M}(X)/L(D) \xrightarrow{\alpha_D} \mathcal{T}[D](X) \xrightarrow{\pi} H^1(D) \to 0,$$

[6.1] *Cokernels* are duals to kernels in category theory. In terms of groups, the cokernel $\operatorname{coker}(\phi)$ for a group homomorphism $\phi : G \to H$ is $H/\operatorname{image}(\phi)$.

[6.2] A sequence of linear transformations

$$U \xrightarrow{\phi} V \xrightarrow{\psi} W$$

between $\mathbb{C}$-vector spaces is said to be *exact* at V if $\ker(\psi) = \operatorname{image}(\phi)$. A longer sequence of maps is exact if it is exact at each interior space.

[6.3] An exact sequence is *short* if it has five spaces and four maps, with the first and last spaces being 0.

where here $\alpha_D : \mathcal{M}(X)/L(D) \to \mathcal{T}[D](X)$ is simply the familiar $\alpha_D : \mathcal{M}(X) \to \mathcal{T}[D](X)$ applied to any element of a coset; it is well defined since $\ker(\alpha_D) = L(D)$.

Suppose now $D_1 \leq D_2$, then we have two short exact sequences associated to each, a truncation map $t = t^{D_1}_{D_2} : \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X)$, and $L(D_1) \subseteq L(D_2)$. Since truncation commutes with the α_D maps, we obtain induced maps between the short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}(X)/L(D_1) & \xrightarrow{\alpha_{D_1}} & \mathcal{T}[D_1](X) & \xrightarrow{\pi_1} & H^1(D_1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & \mathcal{M}(X)/L(D_2) & \xrightarrow{\alpha_{D_2}} & \mathcal{T}[D_2](X) & \xrightarrow{\pi_2} & H^1(D_2) & \longrightarrow & 0 \end{array}$$

where the two squares in the diagram commute. The vertical maps are all onto, so by the Snake Lemma[6.4] we get a short exact sequence for the kernels of these vertical maps (since cokernels of them are all 0):

$$0 \to \ker(\phi) \to \ker(t) \to \ker(\psi) \to 0.$$

Firstly, $\ker(\phi)$ is simply $L(D_2)/L(D_1)$; therefore,

$$\dim \ker(\phi) = \dim L(D_2) - \dim L(D_1).$$

Recall that $L(D)$ is always finitely-dimensional due to Proposition 5.2.6.

Secondly, $\ker(t)$ is the space of those Laurent tail divisors $\sum_p r_p \cdot p$ such that the top term of r_p has degree less than $-D_1(p)$ (for it to be in $\mathcal{T}[D_1](X)$) and the bottom term has degree at least $-D_2(p)$ (for it to be in the kernel of t). Thus we have $D_2(p) - D_1(p)$ possible monomials in z_p which are allowed to appear for p; the conditions at each p are independent, so in total we have

$$\sum_p (D_2(p) - D_1(p)) = \deg(D_2) - \deg(D_1)$$

degrees of freedom, which is the dimension of $\ker(t)$:

$$\dim \ker(t) = \deg(D_2) - \deg(D_1).$$

Finally, let us denote $\ker(\psi)$ by $H^1(D_1/D_2)$. We have a short exact sequence of kernels

$$0 \to L(D_2)/L(D_1) \to \ker(t) \to H^1(D_1/D_2) \to 0,$$

so $H^1(D_1/D_2)$ is finite-dimensional. Indeed, we can easily calculate its dimension.

[6.4] The *Snake Lemma* says, given a commutative diagram like the one we have, there is an exact sequence relating the kernels and cokernels of the vertical maps.

Lemma 6.3.2. *Suppose that D_1 and D_2 are ordinary divisors on a compact Riemann surface X with $D_1 \leq D_2$, then*

$$\dim H^1(D_1/D_2) = [\deg(D_2) - \dim L(D_2)] - [\deg(D_1) - \dim L(D_1)].$$

We shall skip the proof of the following result; however, the proof is where the Laurent Tail Approximation Lemma comes in, so we note that the following results apply only to algebraic curves, at least for the moment:

Proposition 6.3.3. *For any divisor D on an algebraic curve X, $H^1(D)$ is a finite-dimensional $\mathbb{C}$-vector space.*

This allows us to write

$$\dim H^1(D_1/D_2) = \dim H^1(D_1) - \dim H^1(D_2).$$

Plugging back into Lemma 6.3.2 we see that

$$\dim L(D_1) - \deg(D_1) - \dim H^1(D_1) = \dim L(D_2) - \deg(D_2) - \dim H^1(D_2)$$

when $D_1 \leq D_2$. When D_1 and D_2 are not necessarily ordered, we can find a common maximum D and apply $D_1 \leq D$ and $D_2 \leq D$; thus, we have shown that the quantity

$$\dim L(D) - \deg(D) - \dim H^1(D)$$

is constant over all divisors D.

Intuitively, this says that the larger $\deg(D)$ is, the larger the space $L(D)$ can be. This makes sense since $-D$ imposes conditions on zeroes and poles of the meromorphic functions in $L(D)$, and a larger $\deg(D)$ means a loosening of the constraints. Also, the larger $H^1(D)$ is, the larger the space $L(D)$ can be. This also makes sense since a larger $\dim H^1(D) = \dim \operatorname{coker}(\alpha_D)$ roughly says that more functions can have coinciding images under α_D, thus allowing for more functions in $L(D)$.

In particular, we can calculate this quantity by plugging in the zero divisor. Since $\dim L(0) = 1$ (recall that $L(0) \cong \mathbb{C}$) and $\deg(0) = 0$, we have the following formula.

The Riemann-Roch Theorem: Form I. *Let D be a divisor on an algebraic curve D. Then*

$$\dim L(D) - \dim H^1(D) = \deg(D) + 1 - \dim H^1(0).$$

However, this is not very useful yet since it merely shifts the burden of calculating $\dim L(D)$ to calculating $\dim H^1(D)$ and $\dim H^1(0)$, which are also directly related to the existence of meromorphic functions. However, since both are H^1 spaces, we will get a much more powerful result once we identity H^1 spaces with something we've been familiar with.

6.4. **The Residue Map and Serre Duality.** We have been interested in the Mittag-Leffler Problem, asking whether a Laurent tail divisor Z can be the truncation of some meromorphic function f on a compact Riemann surface X. We have seen that $H^1(D)$, the cokernel of map α_D, measures this problem: Z is such a truncation iff its coset in $H^1(D)$ is 0. However, the Residue Theorem gives us another measure. Suppose that we're given a Laurent tail divisor $Z \in \mathcal{T}[0](X)$, and there is a meromorphic function f on X such that $\alpha_0(f) = Z = \sum r_p \cdot p$. Given any holomorphic 1-form ω on X, $f\omega$ can only have poles at the poles of f, and locally the negative terms of the Laurent series for $f\omega$ is only determined by the negative terms of f, which are exactly r_p. Thus,

$$\alpha_0(f\omega) = \sum_p r_p\omega \cdot p.$$

Now since the Residue Theorem states that

$$\sum_p \mathrm{Res}_p(f\omega) = 0,$$

we must have

$$\sum_p \mathrm{Res}_p(r_p\omega) = 0,$$

which poses a necessarily condition on Z for it to be in image(α_0).

The Serre Duality Theorem states that these conditions, suitably generalized for any divisor D, are necessarily and sufficient for the existence of the function f; moreover, note that we may give different ω's, and Serre Duality Theorem states that linearly independent ω's give independent conditions. Therefore, the space $H^1(D)$ can be identified with a space of 1-forms. However, we will see that it's not the space of all meromorphic 1-forms, but those that are sufficiently bounded by the divisor D; in specific, the space

$$L^{(1)}(-D) = \{\omega \in \mathcal{M}^{(1)}(X) : \mathrm{div}(\omega) \geq D\}.$$

To generalize the idea to cases where $D \neq 0$, we start with an arbitrary divisor D on X and a meromorphic 1-form ω in the space $L^{(1)}(-D)$. By definition $\mathrm{div}(\omega) \geq D$, i.e. $\mathrm{ord}_p(\omega) \geq D(p)$ for all p. Therefore we are justified to write

$$\omega = \left(\sum_{n=D(p)}^{\infty} c_n z_p^n \right) \mathrm{d}z_p$$

in local coordinates z_p near p for every p.

Next suppose that f is a meromorphic function on X. Write $f = \sum_k a_k z_p^k$ near p, then the residue of $f\omega$ at p is

$$\operatorname{Res}_p(f\omega) = \text{coefficient of } z_p^{-1}\,\mathrm{d}z_p \text{ in } \left(\sum_k a_k z_p^k \cdot \sum_{n=D(p)}^{\infty} c_n z_p^n\right) \mathrm{d}z_p$$

$$= \sum_{n=D(p)}^{\infty} c_n a_{-1-n}.$$

Note that the residue only depends on the coefficients a_i for f where $i < -D(p)$, which is completely encoded by the Laurent tail divisor $\alpha_D(f)$. This Laurent tail divisor lies in $\mathcal{T}[D](X)$, so the computation of the residue is actually a map

$$\operatorname{Res}_\omega : \mathcal{T}[D](X) \to \mathbb{C} \text{ for } \omega \in L^{(1)}(-D),$$

defined by

$$\operatorname{Res}_\omega\left(\sum_p r_p \cdot p\right) = \sum_p \operatorname{Res}_p(r_p\omega).$$

We call this map a residue map index by ω.

We have just seen that

$$\sum_p \operatorname{Res}_p(f\omega) = \operatorname{Res}_\omega(\alpha_D(f)) \text{ when } \omega \in L^{(1)}(-D),$$

which is always 0 by the Residue Theorem. In other words, $\operatorname{Res}_\omega$ vanishes on all Laurent tail divisors $Z \in \mathcal{T}[D](X)$ that *are* truncations of meromorphic functions, i.e., on $\operatorname{image}(\alpha_D)$. $\operatorname{Res}_\omega$ is linear and vanishes on $\operatorname{image}(\alpha_D)$, so its image of Z is determined on the coset of Z in $\mathcal{T}[D](X)/\operatorname{image}(\alpha_D)$, which is exactly $H^1(D)$. Thus, $\operatorname{Res}_\omega$ descends to a linear map

$$\operatorname{Res}_\omega : H^1(D) \to \mathbb{C},$$

which is an element of the dual space[6.5] $H^1(D)^*$. Since $\omega \in L^{(1)}(-D)$ was fixed until now and $\operatorname{Res}_\omega : H^1(D) \to \mathbb{C}$ is a linear map determined

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[6.5] The *dual space* of a linear space V, denoted by V^*, is the space of all linear functions on V (i.e. functions from V to its base field). V^* is a linear space by pointwise addition and scalar multiplication.

A key result is that when V is finite-dimensional with $\dim V = n$ and a basis $\mathcal{B} = \{v_1, \cdots, v_n\}$, then the set $\mathcal{B}^* = \{v_1^*, \cdots, v_n^*\}$ is a basis of V^*, where v_i^* is a linear function on V defined by its image on $\mathcal{B}$:

$$v_i^*(v_j) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus $\dim V^* = \dim V$ when V is finite-dimensional.

by ω, we have obtained a linear map, also called the *residue map*,

$$\mathrm{Res} : L^{(1)}(-D) \to H^1(D)^*,$$

sending $\omega \in L^{(1)}(-D)$ to the linear function Res_ω on $H^1(D)$.

Remark 6.4.1. The residue map Res is actually the generalization of the discussion we had in the case when $D = 0$. Given a Laurent tail divisor $Z \in \mathcal{T}[D](X)$, and a meromorphic 1-form $\omega \in L^{(1)}(-D)$, the necessary condition

$$\sum_p \mathrm{Res}_p(Z(p)\omega) = 0$$

for Z to be in $\mathrm{image}(\alpha_D)$ is exactly the statement that $\mathrm{Res}_\omega(Z) = 0$, or that the coset $Z + \mathrm{image}(\alpha_D)$ in $H^1(D)$ is in the kernel of Res_ω, which goes from $H^1(D) \to \mathbb{C}$. The statement that the above conditions are necessarily and sufficient translates to:

$$\mathrm{Res}_\omega(Z + \mathrm{image}(\alpha_D)) = 0$$

for all $\omega \in L^{(1)}(-D)$ iff $Z \in \mathrm{image}(\alpha_D)$, i.e. iff the coset $Z + \mathrm{image}(\alpha_D)$ is 0 in $H^1(D)$.

The Serre Duality Theorem states that Res is an isomorphism.

Theorem 6.4.2 (Serre Duality)**.** *For any divisor D on an algebraic curve, the map*

$$\mathrm{Res} : L^{(1)}(-D) \to H^1(D)^*$$

is an isomorphism between $\mathbb{C}$-vector spaces. In particular, for any canonical divisor K on X,

$$\dim H^1(D) = \dim L^{(1)}(-D) = \dim L(K - D)^{6.6}.$$

[6.6] The last equality comes from Proposition 5.2.4.

6.5. **The Equality of Three Genera and The Riemann-Roch Theorem.** A first application of the Serre Duality Theorem is to compute the term $H^1(0)$ in the first form of the Riemann-Roch Theorem. We fix a canonical divisor K on an algebraic curve X of genus g, then

$$\deg(K) = 2g - 2$$

by Proposition 5.1.6. Next, applying Serre Duality to K, we see that

$$\dim H^1(K) = \dim L(K - K) = \dim L(0) = 1.$$

Lastly, we have

$$\dim H^1(0) = \dim L(K).$$

Applying the first form of the Riemann-Roch Theorem, we have

$$\dim L(K) - \dim H^1(K) = \deg(K) + 1 - \dim H^1(0),$$

and using the previous relations we deduce that

$$\dim H^1(0) = \dim L^{(1)}(0) = \dim L(K) = g.$$

The term $\dim H^1(0)$ is sometimes referred to as the *arithmetic genus* of X. The space $L^{(1)}(0)$ is exactly the space $\Omega^1(X)$ of global holomorphic 1-forms on X. The dimension of this space is essentially an analytic invariant, so the term $\dim L^{(1)}(0) = \dim \Omega^1(X)$ is sometimes called the *analytic genus* of X.

Thus we see that a beautiful and profound result of algebraic curves:

Proposition 6.5.1. *Let X be an algebraic curve. All three genera on X, namely*

- *the topological genus g,*
- *the arithmetic genus* $\dim H^1(0)$*, and*
- *the analytic genus* $\dim \Omega^1(X) = \dim L^{(1)}(0)$

are all equal.

The higher-dimensional generalization of this result is called the Hirzebruch-Riemann-Roch Theorem.

Having identified the dimension of the space $H^1(D)$ and $H^1(0)$, we can replace these terms in the Riemann-Roch Theorem to obtain a much more powerful form:

The Riemann-Roch Theorem: Form II. *Let X be an algebraic curve of genus g. Then for any divisor D and any canonical divisor K, we have*

$$\dim L(D) - \dim L(K - D) = \deg(D) + 1 - g.$$

Equivalently, in the language of complete linear systems,

$$\dim|D| - \dim|K - D| = \deg(D) + 1 - g.$$

6.6. **Quick Applications of Riemann-Roch.** In conclusion, we will present some applications of this powerful theorem. The easier applications comes from the cases where $H^1(D) = L(K-D) = 0$. Notice that, by Lemma 5.2.2, this happens every time the divisor $K - D$ has negative degree, and we can get rid of the term $L(K-D)$ to obtain exact formula for $\dim L(D)$. The straightforward case is the following:

Proposition 6.6.1. *Let D be a divisor on an algebraic curve X of genus g with $\deg(D) \geq 2g-1$, then $H^1(D) = 0$ and*

$$\dim L(D) = \deg(D) + 1 - g.$$

We also notice that Riemann-Roch answers the Mittag-Leffler Problem: if Riemann-Roch holds on compact Riemann surfaces, $\dim L(D) \geq \deg(D) + 1 - g > 0$ whenever $\deg(D) \geq g$, therefore guaranteeing a lot of nonconstant meromorphic functions on X. In fact, we are guaranteed the pool of desirable meromorphic functions of the algebraic curves in the following sense:

Proposition 6.6.2. *If X is a compact Riemann surface which satisfies the Riemann-Roch Theorem for every divisor D, then X is an algebraic curve.*

Proof. First we show that $\mathcal{M}(X)$ separates the points of X. Fix two points p and q on X, and consider the divisor $D = (g+1) \cdot p$. By the Riemann-Roch Theorem, we have that $\dim L(D) \geq \deg(D) + 1 - g = 2$, so there is a nonconstant meromorphic function $f \in L(D)$. This function f must have a pole (all holomorphic functions on X are constants), and poles are only allowed at p. Thus f has a pole at p and nowhere else, in particular q, so f separates p and q.

Now we show that $\mathcal{M}(X)$ separates the tangents of X. Fix a point p on X, and consider the divisors $D_n = n \cdot p$. By Proposition 6.6.1, $\dim L(D_n) = n+1-g$ for large n. Thus there are functions in $L(D_{n+1})$ which are not in $L(D_n)$ for large n, but this means that there are functions f_n with a pole of order exactly n at p and no other poles. The function f_n/f_{n+1} then has a simple zero at p. □

We are also able to start some classification of algebraic curves; by doing so we are classifying complex structures on 2-manifolds. For example, we will show that there is essentially only one complex structure on $\mathbb{S}^2$, by showing that all algebraic curves of genus zero is isomorphic to the Riemann Sphere.

Lemma 6.6.3. *Let X be a compact Riemann surface. Suppose that $\dim L(p) > 1$ for some point $p \in X$, then X is isomorphic to the Riemann Sphere.*

Proof. Since $\dim L(p) > 1$, there is a nonconstant meromorphic function f in $L(p)$. This function must have a simple pole at p and no other poles. Therefore, the associated holomorphic map $F : X \to \mathbb{C}_\infty$ has degree one by considering $\sum_{p \in F^{-1}(\infty)} \mathrm{mult}_p(F)$. By Corollary 3.4.3, F is an isomorphism. □

Proposition 6.6.4. *Let X be an algebraic curve of genus zero, then X is isomorphic to $\mathbb{C}_\infty$.*

Proof. Fix any point $p \in X$. Since any canonical divisor K on X has degree $2g - 2 = -2$, the divisor $K - p$ has degree -3, thus again by Lemma 5.2.2 we have $L(K - p) = 0$. Applying Riemann-Roch to the divisor p, we have

$$\dim L(p) = \deg(p) + 1 - g + \dim L(K - p) = 2.$$

By the previous lemma X is isomorphic to the Riemann Sphere. □

In fact, every genus-one algebraic curve is isomorphic to a complex torus, and every genus-two algebraic curve is hyperelliptic (which we have not defined, so we only mention it here).

Lemma 6.6.3 has another consequence.

Lemma 6.6.5. *The canonical linear system*[6.7] *$|K|$ on an algebraic curve X of genus $g \geq 1$, consisted of all nonnegative canonical divisors on X, is base-point-free.*

Proof. Fix a point $p \in X$. By Proposition 5.3.7, we need to show that $\dim L(K - p) = \dim L(K) - 1 = g - 1$. Since $g \geq 1$, $L(p)$ consists of only the constant meromorphic functions by the contrapositive of Lemma 6.6.3. Thus $\dim L(p) = 1$, and applying Riemann-Roch to the divisor p we have that

$$1 = \dim L(p) = \dim L(K - p) + \deg(p) + 1 - g$$

which gives $\dim L(K - p) = g - 1$. □

[6.7]The complete linear system for any canonical divisor K, since all canonical divisors are linearly equivalent.

6.7. **More Applications of Riemann-Roch: Clifford's Theorem and the Existence of Meromorphic 1-Forms.** So far, all of our applications of Riemann-Roch see $H^1(D) = 0$ and gives a formula of $\dim L(D)$. A divisor D is called a *special divisor* when both $\dim L(D)$ and $\dim H^1(D)$ are nonzero. Riemann-Roch tells us that the difference of the two numbers is $\deg(D) + 1 - g$ and in this case we can get an *inequality* on $\dim L(D)$.

Lemma 6.7.1. *Let D_1 and D_2 be two divisors on a compact Riemann surface X, then*

$$\dim L(D_1) + \dim L(D_2) \leq \dim L(\min\{D_1, D_2\}) + \dim L(\max\{D_1, D_2\}).$$

Proof. By the definition of the space $L(D)$ it's easy to see that

$$L(D_1) \cap L(D_2) = L(\min\{D_1, D_2\})$$

and that

$$L(D_1) + L(D_2) \subseteq L(\max\{D_1, D_2\}).$$

Recall that

$$\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$$

for subspaces W_1 and W_2 of a vector space, and hence the result. □

We immediately apply this Lemma to D and $K - D$.

Lemma 6.7.2. *Let D be a special divisor on an algebraic curve X of genus g, then*

$$\dim L(D) + \dim L(K - D) \leq g + 1.$$

Proof. Since $\dim L(D) \geq 1$, from Proposition 5.3.5 the complete linear system $|D|$ is nonempty, so we may choose a positive divisor $D_1 \sim D$. Similarly we may choose a positive divisor $D_2 \sim K - D$. We have $\min\{D_1, D_2\} \geq 0$ and $\max\{D_1, D_2\} \leq D_1 + D_2$. If D_1 and D_2 have disjoint support, both inequalities would be equalities and the result would follow.

However, it is not always the case that we may select D_1 and D_2 with disjoint support. Thus we select D_2 arbitrarily from $|K - D|$ as above. We write $|D| = F + |M|$, where F is the fixed part of $|D|$ and $|M|$ has no base points. Thus there is a positive divisor $D_3 \in |M|$ whose support is disjoint from D_2. Moreover, $\dim L(D_3) = \dim L(M) = \dim L(D)$, and

$$\deg(D_3 + D_2) \leq \deg(F + D_3) + \deg(D_2) = \deg(D) + \deg(K - D) = \deg(K).$$

Thus, applying the previous lemma we have

$$\begin{aligned}\dim L(D) + \dim L(K-D) &= \dim L(D_3) + \dim L(D_2)\\ &\leq \dim L(\max\{D_3, D_2\}) + \dim L(\min\{D_3, D_2\})\\ &= \dim L(D_3 + D_2) + \dim L(0)\\ &\leq \dim L(K) + \dim L(0)\\ &= g+1.\end{aligned}$$

□

Riemann-Roch gives the exact difference between $L(D)$ and $L(K-D)$ while the previous lemma bounds their sum; combining the two we obtain Clifford's Theorem which bounds $\dim L(D)$:

Theorem 6.7.3 (Clifford's Theorem)**.** *Let D be a special divisor on an algebraic curve X, then*

$$2\dim L(D) \leq \deg(D) + 2.$$

Equivalently,

$$\dim|D| \leq \frac{1}{2}\deg(D).$$

We've been interested in the existence of certain meromorphic functions: the Mittag-Leffler Problem asks for meromorphic functions with prescribed Laurent tail divisors, and the space $L(D)$ asks for meromorphic functions with bounded poles and zeroes. We have seen that they're related by the Serre Duality between $H^1(D)$, while measures the Mittag-Leffler Problem, and $L(K-D)$. As a final application of Riemann-Roch, we ask the similar question for meromorphic 1-forms: does there exist meromorphic 1-forms with prescribed zeroes and poles, as measured by divisor D? This is measured by $L^{(1)}(D) \cong L(D+K)$ for any canonical divisor K, so it's the familiar problem.

We can actually do better: while prescribing poles (here we only consider simple poles) we can associate residues to each pole and having the meromorphic 1-form achieve the residue there. Of course, the sum of residues must be zero.

Proposition 6.7.4. *Given an algebraic curve X, a finite set of points $\{p_i\}$ on X, and a corresponding set of complex numbers $\{r_i\}$ such that $\sum_i r_i = 0$, there is a meromorphic 1-form ω on X with simple poles at the p_i's, no other poles, and $\mathrm{Res}_{p_i}(\omega) = r_i$ for each i.*

Proof. Firstly, if the genus g of X is zero, X is the Riemann Sphere, and we can write down the 1-form explicitly in this case. In specific.

let $p_1, \cdots, p_n$ be the points on $\mathbb{C}$ in $\{p_i\}$, and there may or may not be a point $p_{n+1} = \infty$. Then the meromorphic 1-form defined by

$$\omega = f(z)\,\mathrm{d}z = \sum_{i=1}^{n} \frac{r_i}{z - p_i}\,\mathrm{d}z$$

for coordinate z centered at zero, and correspondingly

$$\begin{aligned}\omega &= f\left(\frac{1}{w}\right) \cdot -\frac{1}{w^2}\,\mathrm{d}w \\ &= \sum_{i=1}^{n} \frac{r_i w}{1 - p_i w} \cdot -\frac{1}{w^2}\,\mathrm{d}w \\ &= \sum_{i=1}^{n} \frac{r_i}{1 - p_i w} \cdot -\frac{1}{w}\,\mathrm{d}w \\ &= \frac{\sum_i r_i \left(\prod_{j \neq i}(1 - p_j w)\right)}{\prod_i (1 - p_i w)} \cdot -\frac{1}{w}\,\mathrm{d}w\end{aligned}$$

for coordinate w centered at ∞. When there is no $p_{n+1} = \infty$, the sum of residues $\sum_{i=1}^{n} r_i = 0$, which is also the constant term in numerator; we can cancel w and there will be no pole at ∞, so ω has simple poles at each $p_i \in \mathbb{C}$ with residues r_i and no other poles. When there is $p_{n+1} = \infty$, $\sum_{i=1}^{n} r_i = -r_{n+1}$, and ω will have a simple pole at $p_{n+1} = \infty$ with residue r_{n+1}.

When $g \geq 1$, we may apply Lemma 6.6.5, so the canonical linear system is base-point-free, which means that there is a nonnegative canonical divisor K which has none of the p_i's in its support. Let ω_0 be a meromorphic 1-form whose divisor is K; ω_0 is in fact holomorphic since $K \geq 0$, and ω_0 does not have a zero at any p_i. We will find our desired form ω as $f\omega_0$, for a suitable meromorphic function f.

Choose a local coordinate z_i centered at each p_i, and write $\omega_0 = (c_i + z_i g_i)\,\mathrm{d}z_i$ where $g_i(z_i)$ is holomorphic in z_i; moreover $c_i \neq 0$ because p_i is not a zero of ω_0. Consider the Laurent tail divisor Z whose value at p_i is the Laurent tail $(r_i/c_i)z_i^{-1}$ and is 0 away from the p_i's. We consider Z as being in the space $\mathcal{T}[K](X)$, since $K(p_i) = 0$ for all i.

We claim that a global meromorphic function f in $\alpha_K^{-1}(Z)$ would be the desired meromorphic function; that is, $f\omega_0$ would have the prescribed simple poles with prescribed residues and no other poles. First notice that such an f will have no poles at the p_i's and the points in the support of K. At any point q in the support of K, the order of the pole allowed is no more than $K(q) = \mathrm{ord}_q(\omega)$, which is the order of zero of ω_0. Thus the meromorphic 1-form $f\omega_0$ will not have a pole at q. On the other hand, the pole of f at p_i will be simple since the Laurent tail is $(r_i/c_i)z_i^{-1}$, so will be the pole of $f\omega_0$ since ω_0 is nonzero at p_i, and the residue of $f\omega_0$ will be r_i.

Finally, we need to show that such an f exists, which is a Mittag-Leffler Problem. f exists iff the coset of Z is zero in $H^1(K)$. By Serre Duality we have seen that

$$\dim H^1(K) = \dim L(K-K) = \dim L(0) = 1,$$

so there is exactly one linear condition on Z for the function f to exist. We already know one such linear condition: the sum $\sum_i r_i = 0$. Hence this is the only linear condition, and it is sufficient for f to exist. $\square$

There is a more theoretical approach to this problem. First we define the space of *Laurent tail 1-form divisors*, which are defined as the space of Laurent tail divisors $\mathcal{T}[D](X)$ but with additional dz's at appropriate places. We denote this space by $\mathcal{T}^1[D](X)$. Like $\mathcal{T}[D](X)$, there is a natural map α_D^1 from $\mathcal{M}^1(X)$ to $\mathcal{T}^1[D](X)$ whose kernel is $L^{(1)}(D)$.

Choose a meromorphic 1-form ω_0 whose canonical divisor is K, then we can induct a natural map from $\mathcal{T}[K](X)$ to $\mathcal{T}^1[0](K)$ by multiplication by ω_0. The diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L(K) & \xrightarrow{\iota} & \mathcal{M}(X) & \xrightarrow{\alpha_K} & \mathcal{T}[K](X) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega^1(X) & \xrightarrow{\iota} & \mathcal{M}^1(X) & \xrightarrow{\alpha_K^1} & \mathcal{T}^1[0](X) \end{array}$$

has exact rows, and vertical maps are all isomorphisms, given by multiplication by ω_0. Since $H^1(K) = \operatorname{coker}(\alpha_K)$ has dimension $\dim L(K-K) = 1$, so does $\operatorname{coker}(\alpha_K^1)$. But this map sends a meromorphic 1-form to the negative parts of its Laurent series at every point, so there is exactly one linear condition on a Laurent tail 1-form divisor in $\mathcal{T}^1[0](X)$ for it to be the Laurent tails of a meromorphic 1-form. This condition is that the sum of the residues is zero.

References

[Mir97] Rick Miranda. *Algebraic Curves and Riemann Surfaces*. American Mathematical Soc., 1997.