# Grassmannians and Flag Varieties

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## 1 Introduction and a review of projective space and varieties

The Grassmannian is a concept relating to linear algebra and vector spaces, but relates to algebraic geometry through projective space and can be represented as a topological space. Closely related is the concept of *flag varieties*, which can be constructed as products of Grassmannians. This paper assumes prior knowledge of abstract algebra, including ring theory and algebraic geometry. We aim to show, through the concepts introduced in this paper, both how the direct product of Grassmannians generates a flag variety, and how flag varieties can be written as quotients of algebraic groups.

**Definition 1.1** (Projective Space)**.** For a field $k$, the $n$-dimensional projective space $\mathbb{P}^n_k$ is the set of all lines which pass through the origin of the affine space $\mathbb{A}^{n+1}_k$.

To review, the lines that make up projective space are represented as points, since the lines all go through the origin and two points determine a line. Points in $n$-dimensional projective space are therefore represented in the form $(a_1; a_2; \ldots; a_n)$, where the line represented passes through the origin and the point $(a_1; a_2; \ldots; a_n)$.

**Definition 1.2** (Homogeneous Polynomial)**.** A homogeneous polynomial is a polynomial $f(a_0, \ldots, a_n)$ such that $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n)$. Equivalently, a polynomial is homogeneous if each of its monomial term is of the same degree.

Since points in projective space can be represented using infinitely many different points - for example, $(a_1; a_2; \ldots; a_n)$ is equivalent to $(\lambda a_1; \lambda a_2; \ldots; \lambda a_n)$ - it is important that we ensure that only polynomials for which these points are equivalent are considered when defining projective varieties. Homogeneous polynomials are such polynomials.

**Definition 1.3** (Projective Sets)**.** Projective sets are the vanishing sets of families of homogeneous polynomials.

This corresponds with the definition of homogenous polynomials - equivalent points on a certain line which is part of the vanishing set of a polynomial must all vanish with the polynomial.

**Remark 1.4** (Projective Varieties)**.** Projective sets are varieties if they are irreducible.

Similar to algebraic varieties, projective sets are varieties if they are irreducible.

## 2 The Grassmannian

**Definition 2.1** (The Grassmannian)**.** The Grassmannian, denoted $\mathrm{Gr}(k, V)$, is the linear subspace (of dimension $k$) of an $n$-dimensional vector space V.

For example, the Grassmannian $\mathrm{Gr}(1, V)$ is equivalent to the set of all the lines passing through the origin in the $n$-dimensional vector space $V$. Likewise, the Grassmannian $\mathrm{Gr}(2, V)$ is the set of planes passing through the origin in $V$.

**Proposition 2.2** (Projective Varieties)**.** $\mathrm{Gr}(1, V)$ is equivalent to the projective space of dimension $n - 1$.

As defined earlier, projective space consists of the lines which pass through the origin of affine space of one dimension higher, so $\mathrm{Gr}(1, V)$, which is the 1-dimensional linear subspace (or lines) in $V$, is the same as the projective space of $V$. For example, taking the simplest example of $\mathrm{Gr}(1, \mathbb{R}^2)$, we can define a point $m\mathrm{Gr}(1, \mathbb{R}^2)$, where $m$ is a real number, as $\{(x, y) \in \mathbb{R} : y = mx\}$ to get a better sense of what exactly a point is.

**Proposition 2.3** (Projective Varieties)**.** There is a bijection between the Grassmannian $\mathrm{Gr}(1, \mathbb{R}^2)$ and $\mathbb{R} \cup \infty$

This bijection maps line $l$ with slope $m$ to the point $(1, m) \in \mathbb{R}^2$. Lines with slope $\infty$ are then mapped to points with $x = 0$, thus requiring that the bijection includes $\infty$ as well.

**Proposition 2.4.** The Grassmannian $\mathrm{Gr}(1, \mathbb{R}^2)$ can be represented as a manifold, namely a semicircle.

When representing a point with slope $m$ in $\mathrm{Gr}(1, \mathbb{R}^2)$, we can choose any point $(x, y)$ as long as $\frac{y}{x} = m$. Therefore, we can restrict our representation to include points on the circle $y^2 + x^2 - 1$ where $y \geq 0$; points on this semicircle correspond to

**Remark 2.5.** The Grassmannian can be represented as a homogeneous space.

A homogeneous space (in topology) is defined as a topological space or manifold on which a group $G$ acts. The way to define $\mathrm{Gr}(r, V)$ as a group is by representing it as the quotient group of the *general linear group* of $V$, namely $\mathrm{Gr}(r, V) = \mathrm{GL}(V)/H$.

**Theorem 2.6.** $\mathrm{Gr}(r, V) = \mathrm{GL}(V)/H$ where GL represents the general linear group of $V$ and $H$ acts as the stabilizer of one of its subspaces.

The general linear group of a vector space is defined as the set of its automorphisms, also referred to as the set of its (invertible) linear transformations. To review, the stabilizer of an element $x$ in a group is a subgroup consisting of all elements $g \in G$ such that $g \cdot x = x$. The quotient group formed by this is a homogenous space. The elements of the group $\mathrm{GL}(V)/H$ thus are able to act on the manifold (the Grassmannian).

# 3 The Plucker Embedding and Plucker Coordinates

## 3.1 Plucker Coordinates

Briefly, we define, Plucker coordinates, which are a way of defining a line in 3-dimensional projective space using 6 homogeneous points.

The way to define a line in Euclidean space is to either represent it in terms of two distinct points, or a point and a direction.

Given two points $x$ and $y$ on a line, we can construct two vectors: one for direction (or $d = x - y$), and one for the moment, (or $m = x \times y$).

Although neither determines line individually, together we generate 6 points $d_1, d_2, d_3, m_1, m_2, m_3$ corresponding to the weights of the vector, which uniquely determines the line. The Plucker embedding is a generalization of this system.

## 3.2 The Plucker Embedding

It is possible to express a Grassmannian as a subvariety of a projective space, as studied by Julius Plucker, one of the first people to study nontrivial Grassmannians.

**Definition 3.2.1** (The Plucker Embedding)**.** There is a map from $\mathrm{Gr}(k, V)$ to $\mathrm{P}(\bigwedge^k V)$, the projective space of the $k$th wedge power of $v$.

**Definition 3.2.2** (Wedge Power)**.** For a vector space $V$, $\bigwedge^k V = V \wedge V ... \wedge V$ ($n$ times), where $\wedge$ represents the wedge product of two vectors. If $V$ has a basis of $v_1, \ldots, v_n$, then the $k$th wedge power of V has a basis consisting of $\binom{n}{k}$.

We can thus define an injective mapping of $\mathrm{Gr}(k, V)$, mapping each element of the Grassmannian to the projective space of the $k$th wedge power of $V$.

**Theorem 3.2.3.** The Plucker embedding defines an injective mapping to $P(\bigwedge^k V)$, allowing the Grassmannian to be studied in the context of a projective variety.

*Proof.* For an element $\Lambda \in \mathrm{Gr}(k, V)$, we can take a basis for $\Lambda$, say $\{l_1, l_2, \ldots, l_k\}$. The mapping is defined as: $\phi(\Lambda) = [l_1 \wedge l_2 \wedge \ldots \wedge l_k]$. □

# 4 Flag Varieties

**Definition 4.1** (Flags). In linear algebra, a flag is a chain of increasing subspaces of a vector space $V$, each of which is a proper subset of the last.

Each subspace is strictly contained in the next, and the chain starts and ends with $\{0\}$ and $V$. The signature of a flag $\{0\} = A_1 \subset A_2 \subset \ldots \subset A_l = V$ is defined as the sequence $(\dim(A_1),\ \dim(A_2), \ldots \dim(V))$. A flag is called a complete flag if for $\dim V_i = i$ in the sequence $0 = v_0 \subset V_i \ldots \subset V_k = V$ where $k$ is the dimension of $V$, eg. if its signature is of the form $(1, 2, 3, \ldots n)$. If this is untrue, the flag is called partial.

**Definition 4.2** (Flag Varieties). Flag varieties are varieties consisting of flags of a vector space $V$ over a field $F$.

Flag varieties, as discussed later, can be expressed in the form of direct products of Grassmannians. Flag varieties are also projective varieties, a fact which follows from the fact that they are generated by the direct product of Grassmannians.

**Theorem 4.4.** Flag varieties can be written as direct products of Grassmannians.

*Proof.* We take, for any sequence of $l$ integers, with $0 < a_1 < a_2 < \ldots < a_l < n$. We then define $\mathbb{F}(a_1, \ldots, a_l, n)$ to be the set of all flags in $\mathbb{C}^n$ with their signatures corresponding to the sequence $a_n$.

**Remark 4.5** (Projective Varieties). $\mathbb{F}(a_1, \ldots, a_l, n)$ is contained within $\mathrm{Gr}(a_1, V) \times \ldots \mathrm{Gr}(a_n, V)$, because each subspace with dimension $a_k$ has a corresponding element in $\mathrm{Gr}(a_k, V)$. In the case of $l = 1$, $\mathbb{F}(a_1, n) = \mathrm{Gr}(1, V)$.

Now we must show that $\mathbb{F}(a_1, \ldots, a_l, n)$ has the structure of a variety.

To do this, we must show that $\mathbb{F}(a_1, \ldots, a_l, n)$ is a *closed algebraic set*, which we can do by showing that it is a union of closed sets.

To do this, we can define a function $\pi_{ij}$ which, for $0 < i < j \leq l$, is the restriction to $\mathbb{F}(a_1, \ldots, a_l, n)$ of the projection $\mathrm{Gr}(a_1, V) \times \ldots \mathrm{Gr}(a_n, V) \to \mathrm{Gr}(a_i, V) \times \mathrm{Gr}(a_k, V)$. Then, reversing this map, we have:

$$\mathbb{F}(a_1, \ldots, a_l, n) = \bigcap_{i,j} \pi_{ij}^{-1}(\mathbb{F}(a_i, a_j)$$

.

Because this is a closed set, we now know that flag varieties generated in this form are closed algebraic sets and are indeed varieties. □

# 5 Algebraic Groups and Borel Subgroups

Flag varieties can be written as quotients of algebraic groups, which produces a number of interesting results in the realm of algebraic geometry.

**Definition 5.1** (Algebraic Group)**.** An algebraic group is an affine group scheme. A group scheme is a scheme which includes 3 morphisms:

- $m$: $G \times_{\mathbb{C}} G \to G$
- $i$: $G \to G$
- $\epsilon$: $\mathrm{Spec}(\mathbb{C}) \to G$.

These are all morphisms in terms of the algebraic variety (to review, a morphism is a function between two varieties $X$ and $Y$ which is given in terms of a set of polynomials in $X$).

Further, we must define the Borel and parabolic subgroups of any group $G$, and by putting these two pieces together, we are able to see how flag varieties can be generated through algebraic groups.

The definitions of these subgroups are pretty straightforward:

**Definition 5.2** (Borel Subgroup)**.** A Borel subgroup is a subgroup which is maximal out of the set of subgroups which are connected and solvable.

**Definition 5.2.1** (Connected Space)**.** A connected space is a topological space which cannot be represented as the union of two disjoint open subsets.

**Definition 5.3** (Parabolic Subgroup)**.** A parabolic subgroup is a subgroup such that $B/P$ is a projective variety.

All Borel subgroups are parabolic, and both are self-normalizing, but this is not really important to the topics in this paper, so the proof is omitted.

Next, we must define flag varieties in the context of Borel subgroups and algebraic groups.

As defined earlier, a complete flag is one with a signature of $(1, 2, \ldots, n)$. A flag variety with this signature is thus called a full flag variety.

Given an integer $n$, we have the basis of $\mathbb{C}^n$ to be $\{e_1, e_2, \ldots, e_n\}$. We can thus construct a full flag variety by taking the flag $\{0\}, \{e_1\}, \{e_1, e_2\}, \ldots, \{e_1, e_2, \ldots, e_n\}$.

By permuting the basis elements, we observe that the general linear group $GL_n$ can be used to generate full flag varieties (they are not necessarily unique). However, when we assign $B$ to be the upper triangular matrices, we find that $GL_n/B$ aggregates the duplicates and categorizes them.

Therefore we have:

**Theorem 5.4.** $\mathbb{F}(a_1, \ldots, a_l, n) \cong GL_n/B$

based on the reasoning described above. From this fact follows that for each parabolic subgroup, there is a corresponding flag variety $\mathbb{F}(a_1, \ldots, a_l, n)$, and vice versa.

## References


[Ber15] Aaron Bertram. Flag varieties and representations, 2015.

[Fer16] Carlos Eduardo Duran Fernandez. An elementary introduction to the grassmann manifolds, 2016.

[Hud07] Drew A Hudec. The grassmannian as a projective variety, 2007.

[Jia19] Yan-Bin Jia. Plucker coordinates for lines in the space, 2019.

[Sch13] Luca Schaffler. Flag varieties, Dec 2013.


[Sch13] [Ber15] [Fer16] [Hud07] [Jia19]