TROPICAL ALGEBRAIC GEOMETRY

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1. Introduction

Tropical geometry arises from the study of the real numbers in which the usual product is replaced with the sum and the usual sum is replaced with the maximum (or equivalently, the minimum). Polynomials defined using these operations are convex and piecewise linear functions. We can typically analyze the curves associated with these tropical polynomials using combinatorial techniques. Hence, curves in tropical algebraic geometry are much easier to study and understand than classical algebraic curves.

Most concepts from classical algebraic geometry have tropical analogues. Classical objects like lines, polynomials, and curves share many properties with their tropical analogues. However, it is far easier to construct a tropical curve with some desired properties than a classical algebraic curve satisfying an analogue of those properties. After we find the desired object in tropical geometry, we can transform it to an object in classical algebraic geometry, which (hopefully!) has the same desired property. Hence, studying tropical algebraic geometry and its relation to classical algebraic geometry can give us valuable computational and theoretical tools in classical algebraic geometry.

Tropical algebraic geometry is a developing mathematical field and has connections to many computational problems. Since the most basic operation in tropical algebra is the minimum or maximum, it is natural to describe optimization problems using the language of tropical algebra.

In this paper, we seek to introduce the basic concepts of tropical algebraic geometry. The first part of this paper focuses on defining and giving graphical intuition for tropical arithmetic and tropical polynomials. We look at motivations for a tropical analogue of Bézout's theorem, then build up the tools to prove the tropical version of Bézout's Theorem.

2. Tropical Rings and Polynomials

In tropical algebraic geometry, we work over the real numbers with $-\infty$ with different addition and multiplication. For our addition and multiplication operations, we will have $\max(\cdot, \cdot)$ and addition, respectively. The new operations may not satisfy all of the ring axioms. In fact, they don't, since elements generally don't have an additive inverse in this ring. This gives rise to the notion of a *semiring*.

Definition 2.1. A *semiring* is a set R equipped with two binary operations $+$ and $\cdot$, called addition and multiplication, that satisfy the ring axioms except that R does not need to have additive inverses.

Example. The set of non-negative integers $\mathbb{Z}_{\geq 0}$ with the regular addition and multiplication forms a semiring. We can check that $\mathbb{Z}_{\geq 0}$ contains 0 and 1 and that addition and multiplication are commutative, associative, and distributive. However, $\mathbb{Z}_{\geq 0}$ does not contain additive

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tropical semiring	$\mathbb{R}$ with ordinary $(+,\cdot)$	value
$5 \oplus -\infty$	$\max(5, -\infty)$	5
$5 \odot -\infty$	$5 + (-\infty)$	$-\infty$
$23 \oplus 0$	$\max(23, 0)$	23
$23 \odot 0$	$23 + 0$	23
$3 \oplus 4 \oplus 17$	$\max\{3, 4, 17\}$	17
$7 \oplus 3 \odot 6$	$\max(7, 3 + 6)$	9

Table 1. Tropical arithmetic and their equivalent formulations in classical arithmetic

inverse. For instance, there is no element $x \in \mathbb{Z}_{\geq 0}$ such that $3 + x = 0$. Thus, $\mathbb{Z}_{\geq 0}$ is a semiring.

Definition 2.2. The *max tropical semiring* is the semiring $\mathbb{R} \cup \{-\infty\}$ with addition $\oplus$ and multiplication $\odot$ given by

$$x \oplus y = \max(x, y), \quad x \odot y = x + y.$$

Similarly, the *min tropical semiring* consists of the set $\mathbb{R} \cup \{\infty\}$ along with the operations

$$x \oplus y = \min(x, y), \quad x \odot y = x + y.$$

The max tropical semiring is isomorphic to the min tropical semiring under the negation map $x \mapsto -x$, so we only need to study one of them to understand the other. In this paper, we will use the max convention and simply refer to the max tropical semiring as the *tropical semiring.*

In the tropical semiring, the additive identity is $-\infty$ and the multiplicative identity is 0. Some evaluations of expressions in the tropical semiring are given in Table 1 to help the reader get familiar with the arithmetic.

There is an interesting relationship between tropical and classical arithmetic.

Proposition 2.3. *Let $t > 1$ be a parameter and define the arithmetic operations on $\mathbb{R} \cup \{-\infty\}$ by*

$$x +_t y = \log_t(t^x + t^y), \quad x \cdot_t y = \log_t(t^x \cdot t^y).$$

These operations converge to max tropical arithmetic for $t \to \infty$.

Proof. From logarithm rules, it is clear that

$$x \cdot_t y = \log_t(t^x \cdot t^y) = \log_t(t^{x+y}) = x + y.$$

Without loss of generality, suppose $x \leq y$. Then we can factor out t^y from $t^x + t^y$ to obtain

$$x +_t y = \log_t(t^x + t^y) = \log_t(t^y(t^{x-y} + 1)) = y + \log_t(t^{x-y} + 1).$$

As $t \to \infty$, $t^{x-y} \to 0$. Thus, $\log_t(t^{x-y} + 1) \to \log_t(1) = 0$ and we get $x +_t y \to \max(x, y)$ as desired. ■

Now that we have tropical arithmetic, we are ready to define tropical polynomials.

Definition 2.4. Let $x_1, \ldots, x_n$ be variables in the tropical semiring. A *tropical monomial* is a product of these variables, where repetition is allowed. The *degree* of a monomial is the number of variables counting multiplicity.

tropical polynomial	classical evaluation	degree
17	17	0
$x \oplus 3$	$\max(x, 3)$	1
$5 \odot x \oplus 1$	$\max(5 + x, 1)$	1
$7 \odot x^2 \oplus 3 \odot x \oplus 2$	$\max\{7 + 2x, 3 + x, 2\}$	2
$xy \oplus 5$	$\min(x + y, 5)$	2
$xy^2 \oplus 4 \odot x \oplus 6 \odot y$	$\min\{x + 2y, 4 + x, 6 + y\}$	3
$5 \odot x^2yz^2 \oplus 1 \odot wx \oplus 9 \odot y$	$\min\{5 + 2x + y + 2z, 1 + w + x, 9 + y\}$	5

Table 2. Tropical polynomials and their evaluations in classical arithmetic

Using commutativity of multiplication, we can sort the product and write the monomial as a product of powers of the variables. For instance, we can write

$$x_2 \odot x_3 \odot x_1 \odot x_1 \odot x_2 \odot x_3 \odot x_4 \odot x_1 = x_1^3 x_2^2 x_3^2 x_4$$

where we recognize that the operation between the powers of variables is multiplication in the semiring i.e., $\odot$. We can evaluate this monomial in classical arithmetic and find that it represents the linear function $3x_1 + 2x_2 + 2x_3 + x_4$.

Definition 2.5. A *tropical polynomial* is a finite linear combination of tropical monomials:

$$p(x_1, x_2, \ldots, x_n) = a \odot x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \oplus b \odot x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \oplus \cdots . \tag{2.1}$$

The *degree* of a polynomial is the maximum degree of the monomials whose coefficient is not $-\infty$.

For classical polynomials, if a monomial were to have a coefficient of 0, the monomial would always evaluate to 0, and because 0 is the additive identity, the monomial with 0 as its coefficient wouldn't contribute anything to the value of the polynomial. Thus, when we write $x^2 + 1$, we implicitly realize that the coefficient of x is 0. Similarly, $-\infty$ is the additive identity in tropical arithmetic, so a tropical monomial with $-\infty$ as its coeffiicient has no chance of making any contribution to the polynomial. Hence, when we write $x^2 \oplus 1$, we must realize that the coefficient of x is $-\infty$. This is why we do not look at monomials with $-\infty$ as their coefficient when calculating the degree of a tropical polynomial.

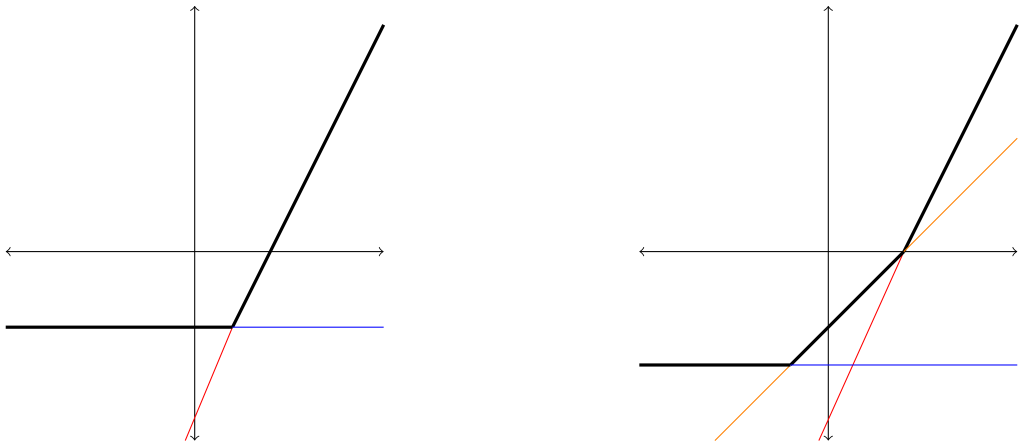
Similarly, when we write $x^2 \oplus 5$, the coefficient of x^2 is implicitly 0, the multiplicative identity in tropical arithmetic.

In Table 2, we express some tropical polynomials in terms of classical arithmetic to get a feel for how tropical polynomials evaluate in classical arithmetic. From Table 2, we see that the tropical polynomial $p(x_1, x_2, \ldots, x_n)$ from Equation (2.1) represents the function from $\mathbb{R}^n$ to $\mathbb{R}$ given by

$$p(x_1, x_2, \ldots, x_n) = \max\{a + i_1x_1 + \cdots + i_nx_n, b + j_1x_1 + \cdots + j_nx_n, \ldots\}.$$

Hence, we conclude that $p(x_1, x_2, \ldots, x_n)$ is a piecewise linear function with the following properties:

- p is continuous,
- p is piecewise linear with finitely many pieces, and
- p is convex: $p(\frac{1}{2}(\boldsymbol{x} + \boldsymbol{y})) \leq \frac{1}{2}(p(\boldsymbol{x}) + p(\boldsymbol{y}))$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.



(a) Two linear pieces **(b)** Three linear pieces

Figure 1. The two possible graphs of a tropical quadratic univariate polynomial

3. Graphs of Tropical Polynomials

In this section, we will try to gain intuition for the graphs of tropical polynomials. To start, we consider a quadratic polynomial in one variable $a \odot x^2 \oplus b \odot x \oplus c$. In classical arithmetic, this is a piecewise function with at most three linear parts: $\max\{a + 2x, b + x, c\}$. Why is it *at most* three and not exactly three? We can deduce that as $x \to \infty$, the polynomial will take on the value $a + 2x$, and as $x \to -\infty$, the polynomial will take on the value c. We do not know whether there is a $x_0 \in \mathbb{R}$ such that $a + 2x_0 < b + x_0 > c$. That is, we do not know whether the graph will have the third linear piece. If b is big enough, then there will be three pieces. Otherwise, there will be two. Figure 1 depicts the two possibilities for the graph of $a \odot x^2 \oplus b \odot x \oplus c$.

Notice how the graph, in stages, gets steeper out as $x \to \infty$. Intuitively, this is what it means for tropical polynomials to be convex functions.

More generally, if we graph a degree d tropical polynomial in one variable, we would first draw d lines on the planes with slopes $1, 2, 3, \ldots, d$, then find which of the lines have the greatest y-value at each x-value. We would get something that looks very similar to Figure 1.

Now let's consider polynomials in two variables. Now we must graph planes in three dimensions then for each (x, y), we find the plane that has the largest z-value. Example graphs for bivariate tropical polynomials are shown in Figure 2.

Notice that the "edges" of the graph of a polynomial in two variables are line segments and half-rays. This is a direct consequence of them being the intersections of planes.

It is a difficult to draw out a 3D graph every time we want to visualize a tropical polynomial in two dimensions, so instead we can draw out those aforementioned "edges" of the graph and write in what value the graph takes on in each region of the (x, y) plane. Figure 3 demonstrates how we can represent a bivariate polynomial in a 2D figure.

We consider these intersections of planes as the "roots" of p. Much like how we defined the vanishing set of a polynomial in algebraic geometry as the set of zeroes of the polynomial, we can define the *hypersurface* $V(p)$ of p as the "roots" of p. More formally:

Definition 3.1. The set of points where a tropical polynomial p is non-differentiable is called the associated *hypersurface* $V(p)$. In two variables, it is equivalently the set of points (x_0, y_0)

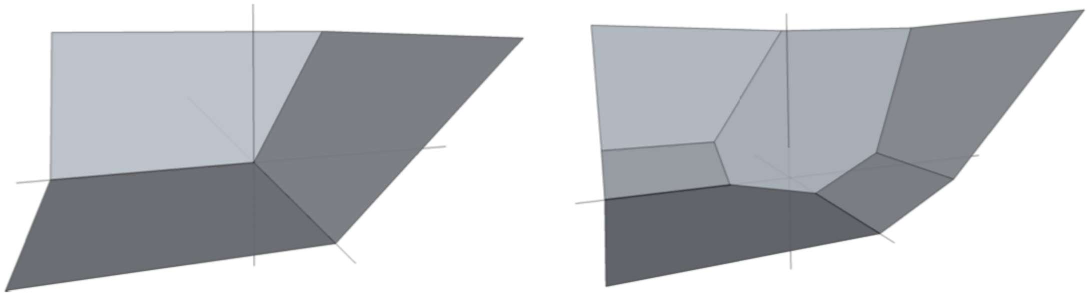


Figure 2. The graphs of two polynomials in two variables. The plot on the left shows the linear polynomial $0 \oplus x \oplus y$. On the right hand side, we see the graph of the quadratic polynomial $0 \oplus x \oplus y \oplus (-1) \odot x^2 \oplus 1 \odot xy \oplus (-1) \odot y^2$.

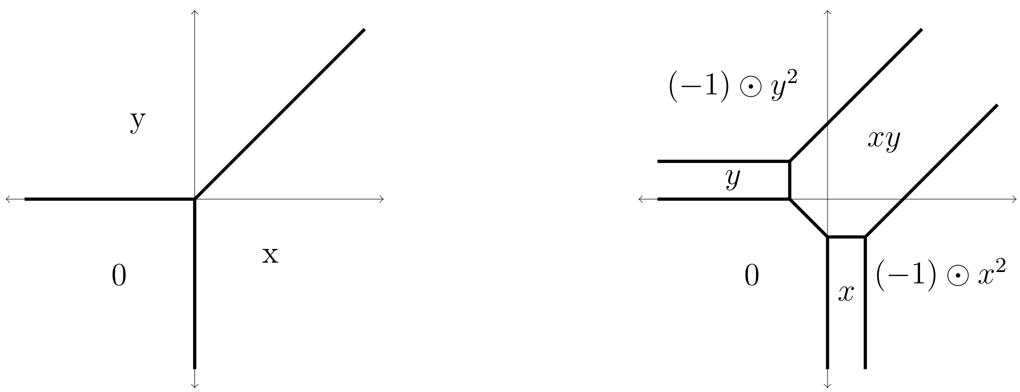


Figure 3. The "edges" of the graphs from Figure 2

such that there exist pairs $(i, j) \neq (k, l)$ satisfying $p(x_0, y_0) = a_{i,j} + ix_0 + jy_0 = a_{k,l} + kx_0 + ly_0$, and $a_{i,j}x^iy^j$ and $a_{k,l}x^ky^l$ are monomial terms of $p(x, y)$.

When $V(p)$ has dimension 1, we call it a *tropical curve*. When $V(p)$ has dimension 2, it would be a surface, and so on. So far, all of the hypersurfaces of bivariate polynomials that we saw had dimension 1.

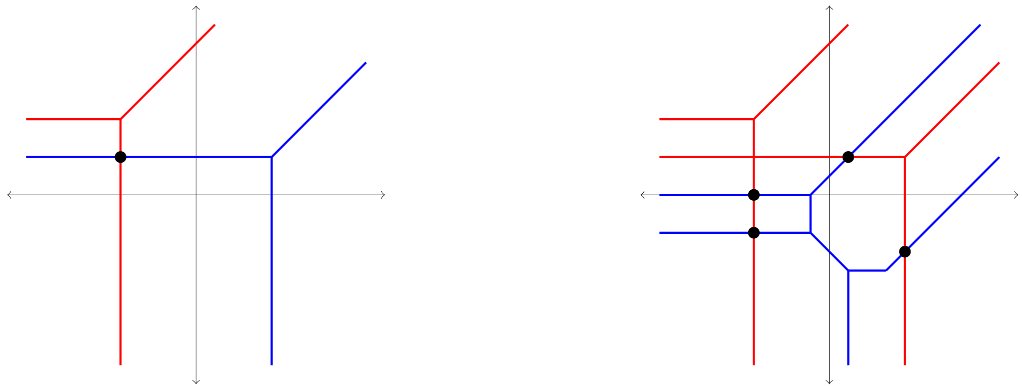
4. Tropical Analogue of Bézout's Theorem for Two Variables

In this section, we look at motivating examples for Bézout's theorem in tropical algebraic geometry. For concreteness, we will only look at tropical curves i.e., work in two variables x, y, in this section.

The lines in tropical geometry are curves defined by the linear polynomial $a \oplus i \odot x \oplus j \odot y$. These may seem nothing like lines, but they actually preserve many of the properties of lines from classical geometry:

- Two tropical lines either intersect at infinitely many points or just one point.
- For most choices of a pair of points, there exists a unique tropical line that passes through both points.

Figure 4a gives an example of the first property in action. We leave it to the reader to convince themselves that the second property holds.



(a) 2 lines: 1 point of intersection. **(b)** 2 quadratics: 4 points of intersection.

Figure 4. Intersections of tropical curves.

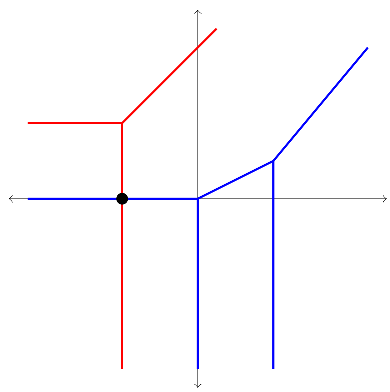


Figure 5. There is only 1 point of intersection of the curves defined by the line $0 \oplus 2 \odot x \oplus (-2) \odot y$ (red) and the quadratic $0 \oplus x \oplus y \oplus y^2 \oplus (-2) \odot x^2$ (blue).

In fact, the classical property that a degree c curve and a degree d curve generally intersect at cd points also generally holds in the tropical case. Figure 4b gives an example of two quadratic curves intersecting at 4 distinct points.

However, there are instances in which we get fewer than cd intersections, as in Figure 5. Why is this case different than the previous two examples? Remember that in the classical Bézout's theorem, some intersections were counted multiple times. We must define a similar notion for the tropical case. Actually, we will define intersection multiplicity later later, after we introduce some new ideas that make the proof of Bézout's theorem easier.

We can actually do better than the classical Bézout's Theorem by carefully handling the case when the curves have non-traverse intersections i.e., one of the curves passes through a vertex of the other curve (See Figure 6a, 6b). Say we have two curves C, D that intersect at infinitely many points. Let C_ε and D_ε be translations of C and D by some small distance $\varepsilon > 0$. If we choose the direction of our translations correctly, we can get finitely many intersections between C_ε and D_ε. In fact, no matter the choice of C_ε and D_ε (as long as they have finitely many intersections), the points of intersection converge when we let $\varepsilon \to 0$. The limits actually lie on $C \cap D$. Thus, instead of thinking about the infinite intersection of C and

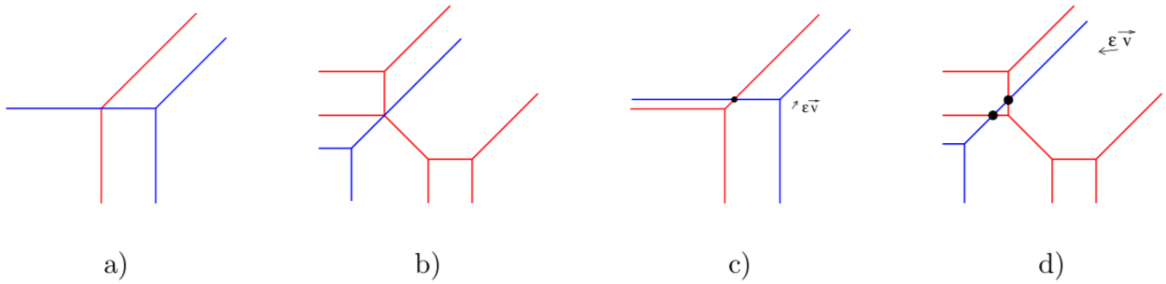


Figure 6. Non-traverse intersections and a translation

D, we can think about the finite intersection that comes from $\lim_{\varepsilon\to 0}(C_\varepsilon \cap D_\varepsilon)$. See Figure 6 for an example of this limiting process.

We refer to this intersection property of tropical curves as the *stable intersection principle.* Although the following theorem on stable intersection is too difficult for us to prove in this paper, it will allow us to state the complete tropical version of Bézout's theorem for the case of two variables.

Theorem 4.1 (Stable Intersection Principle). *Let C and D be two tropical curves such that they do not intersect traversally. Let C_ε and D_ε be two nearby curves such that they intersect traversally. The limit of the point figurations $C_\varepsilon \cap D_\varepsilon$ is independent of the choice of perturbations. It is a well-defined multiset of cd points contained in the intersection $C \cap D$. We call this limit the* stable intersection *of the curves C and D. This is a multiset of points, denoted by*

$$C \cap_{st} D = \lim_{\varepsilon\to 0}(C_\varepsilon \cap D_\varepsilon).$$

Thus we obtain Bézout's theorem:

Theorem 4.2 (Bézout). *Any two tropical curves of degrees c and d in $\mathbb{R}^2$, no matter how special they might be, intersect stably in a well-defined multiset of cd points.*

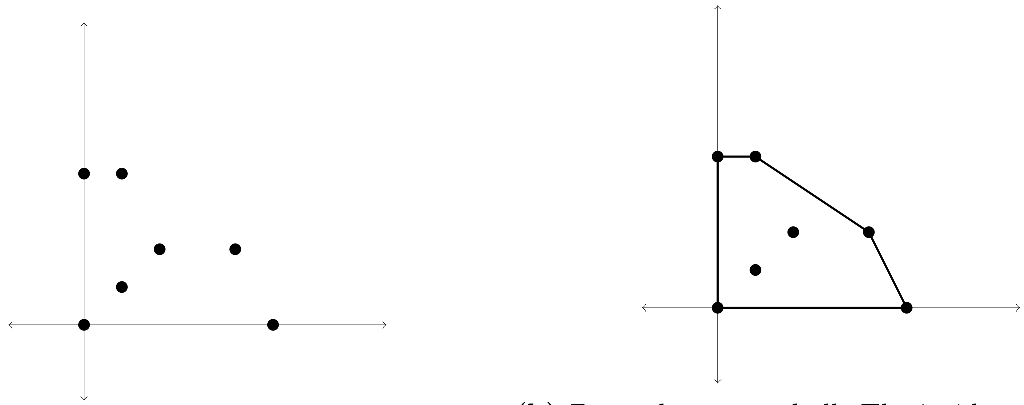
The proof of this theorem is not too difficult with the theory of Newton polygons and dual subdivisions of tropical curves.

5. Newton Polygons and Dual Subdivisions

Definition 5.1. Let $p(x, y)$ be a polynomial in two variables, in either classical or tropical arithmetic. Its *Newton polygon* $\mathrm{Newt}(p)$ is defined as the convex hull in $\mathbb{R}^2$ of all points (i, j) such that $x^i y^j$ appears in the expansion of $p(x, y)$.

The Newton polygon is very easy to draw given the equation of the polynomial. First, plot all of the points (i, j) such that $ax^i y^j$ is a monomial of the polynomial for some $a \in \mathbb{R}$. The convex hull is drawn by connecting the "outermost" points so that the resulting polygon is convex and contains all of the previously drawn points (i, j). The Newton polygon is the convex hull of these points. Figure 7 shows the process of drawing the Newton polygon for the polynomial $x^5 + 3x^4y^2 + 2xy^4 + y^4 + x^2y^2 + x + 7y + 2$.

The tropical curve C of a tropical polynomial p is closely related to a subdivision of its Newton polygon $\mathrm{Newt}(p)$. We draw the subdivision as follows: for each vertex $v = (x_0, y_0)$ of C, we associate it with the convex hull Δ_d of the points (i, j) such that $p(x_0, y_0) = a_{i,j} +$



(a) Plot (i, j) for every nonzero monomial $x^i y^j$. **(b)** Draw the convex hull. The inside points are excluded from the final polygon.

Figure 7. Drawing the Newton polygon of
$x^5 + 3x^4y^2 + 2xy^4 + y^4 + x^2y^2 + x + 7y + 2$.

(a) $0 \oplus x \oplus y$ **(b)** $0 \oplus x \oplus y \oplus (-1) \odot x^2 \oplus 1 \odot xy \oplus (-1) \odot y^2$

Figure 8. Subdivisions dual to the curves from Figure 3

$ix_0 + jy_0$. This convex hull is contained inside $\mathrm{Newt}(p)$. The fact that $p(x, y)$ is a convex piecewise linear function implies that the collection of Δ_d is a subdivision of $\mathrm{Newt}(p)$. This subdivision contains almost complete information about the tropical curve, so we call it the *dual subdivision* of C. That is, the tropical curve is determined by its dual subdivision up to translation and choice of lengths of its edges. The dual subdivision of the polynomials from Figure 2 and 3 are shown in Figure 8. For some more involved dual subdivisions, see Figure 9.

The union of two tropical curves C_1 and C_2 is related to the product of two polynomials defining the two curves. This is analogous to the statement $V(IJ) = V(I) \cup V(J)$ from classical algebraic geometry.

Proposition 5.2. *Let C_1 and C_2 be tropical curves, described by the tropical polynomials $p_1(x, y)$ and $p_2(x, y)$ respectively, that intersect in a finite collection of points away from the vertices of both curves. The union $C_1 \cup C_2$ is defined by the tropical polynomial $p_1(x, y)p_2(x, y)$. As a consequence, the degree of $C_1 \cup C_2$ is the sum of the degrees of C_1 and C_2.*

Proof. Let's think about what the curve defined by $p_1(x, y)p_2(x, y)$ looks like. In classical arithmetic, the polynomial is just the sum of two max functions, each given by $p_1(x, y)$ and $p_2(x, y)$. Whenever $p_1(x, y)p_2(x, y)$ is non-differentiable, $p_1(x, y)$ or $p_2(x, y)$ must be non-differentiable. The converse also holds. This implies that (x, y) is on the curve defined by $p_1(x, y)p_2(x, y)$ if and only if it is on at least one of C_1 and C_2. This exactly describes the

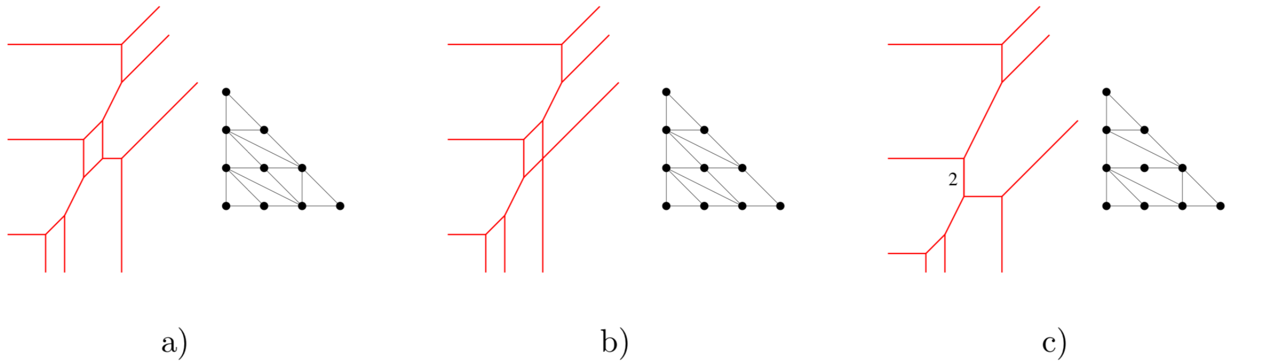


Figure 9. More complex tropical curves and their respective dual subdivisions

condition for a point to be on the curve $C_1 \cup C_2$. Thus, we have proved that $C_1 \cup C_2$ is the tropical curve defined by the polynomial $p_1(x, y)p_2(x, y)$. ■

Now, we can think about what the dual subdivision of $p_1(x, y)p_2(x, y)$ says about the intersection points of C_1 and C_2. An inner polygon in any dual subdivision corresponds to a vertex of the curve and the edges radiating out of that vertex. If two intersecting edges of $C_1 \cup C_2$ both belong to C_1 or C_2, then the corresponding polygon in the dual subdivision of $C_1 \cup C_2$ is exactly the same polygon from the dual subdivision of C_1 or C_2. Thus, there is essentially a copy of the dual subdivision of C_1 and C_2 inside the dual subdivision of $C_1 \cup C_2$.

If two intersecting edges of $C_1 \cup C_2$ belong to both C_1 and C_2, then we get a new polygon in the dual subdivision of $C_1 \cup C_2$ that wasn't in the dual subdivision of C_1 or C_2. We define the the multiplicity of the intersection point of two edges of C_1 and C_2 to be the area of the polygon corresponding to the vertex in $C_1 \cup C_2$.

Definition 5.3. Let C_1 and C_2 be two tropical curves which intersect in a finite number of points and away from the vertices of the two curves. If p is a point of intersection of C_1 and C_2, the tropical multiplicity of p as an intersection point C_1 and C_2 is the area of the parallelogram dual to p in the dual subdivision of $C_1 \cup C_2$.

We can see this definition of multiplicity in action in Figure 10. In the first and second examples in the figure, all of the intersections have multiplicity 1. This agrees with the expectation that we have 1 and 2 points of intersection, respectively. In the third example, we only see 1 point of intersection but we want 2 points of intersection since the curves are defined by a line and a quadratic. Definition 5.3 fixes this discrepancy by giving the point of intersection multiplicity 2 (because the area of the central parallelogram is 2).

We remark that there is another way of defining intersection multiplicity, which takes into consideration the properties of the curves independently and the fundamental vectors (integer vectors $\langle x, y\rangle$ parallel to the edges that have $\gcd(x, y) = 1$) of the edges that intersect and finding a determinant-like expression that depends on both edges. This definition may be better motivated than Definition 5.3 and give better insight on why Bézout's theorem should work in the tropical case. However, I have chosen to use Definition 5.3 as the definition for intersection multiplicity as it easily allows us to prove Bézout's theorem. Also, I wasn't able to find a good source material elaborating on this more involved definition for intersection multiplicity.

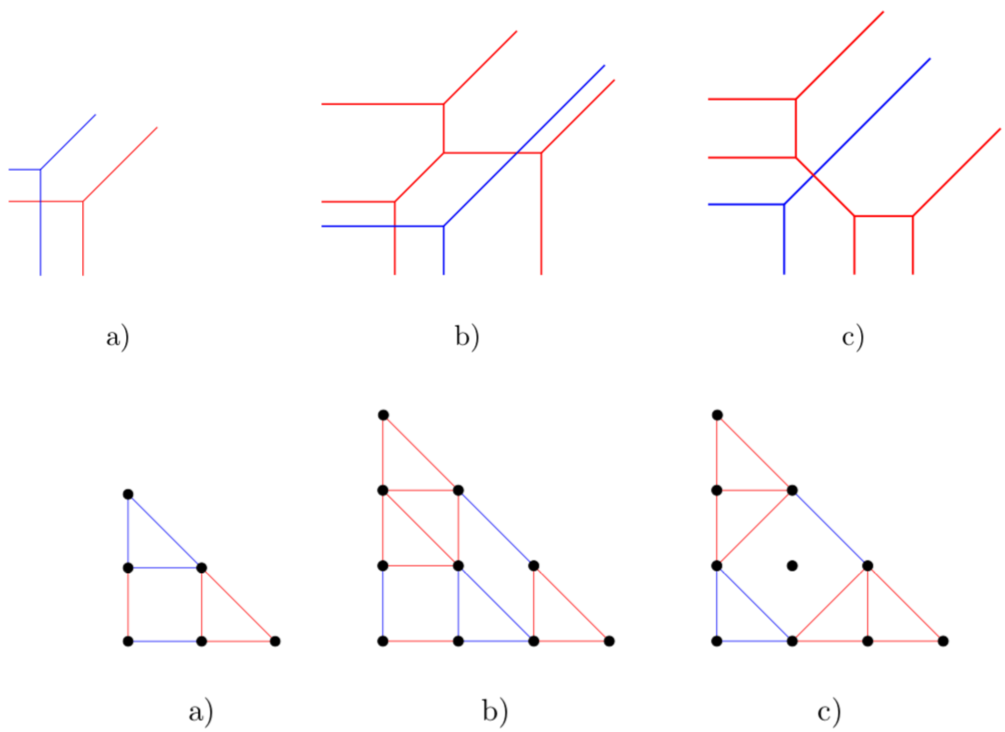


Figure 10. Intersections of curves and the subdivisions dual to the union of the curves

Anyway, with this machinery of dual subdivisions and intersection multiplicity, we are now ready to prove Bézout's theorem!

6. Proof of the Tropical Bézout's Theorem

Theorem 6.1 (Bézout)**.** *Any two tropical curves of degrees c and d in $\mathbb{R}^2$ that intersect traversally intersect in a well-defined multiset of cd points.*

Proof. Let C_1 and C_2 be tropical curves of degree d_1 and d_2, respectively. For simplicity, we will only prove the theorem for when $a_{0,0}, a_{d_1,0}, a_{0,d_1} \neq -\infty$ in $p_1(x, y)$ and similarly for $p_2(x, y)$. Equivalently, we are only considering tropical polynomials whose Newton polygons are triangles with vertices at $(0, 0), (d, 0), (0, d)$ where d is the degree of the polynomial.

Let s be the sum of the multiplicities of the points in the intersection. Consider the dual subdivision of $C_1 \cup C_2$. The polygons of the subdivisions fall into three categories: those dual to a vertex of C_1 having a total area of $\frac{1}{2}d_1^2$, those dual to a vertex of C_2 having total area $\frac{1}{2}d_2^2$, and those dual to an intersection point of C_1 and C_2. Since the curve $C_1 \cup C_2$ is of degree $d_1 + d_2$, $\mathrm{Newt}(p_1 p_2)$ is a triangle of area $\frac{1}{2}(d_1 + d_2)^2$, and hence the sums of the areas of these polygons is equal to $\frac{1}{2}(d_1 + d_2)^2$. Therefore,

$$\frac{1}{2}d_1^2 + \frac{1}{2}d_2^2 + s = \frac{1}{2}(d_1 + d_2)^2.$$

Solving for s gives

$$s = \frac{(d_1 + d_2)^2 - d_1^2 - d_2^2}{2} = d_1 d_2$$

as desired. ■

Using the Stable Intersection Principle (Theorem 4.1), we conclude that

Theorem 6.2 (Bézout)**.** *Any two tropical curves of degrees c and d in $\mathbb{R}^2$, no matter how special they might be, intersect stably in a well-defined multiset of cd points.*