LEFSCHETZ HYPERPLANE THEOREM

SRITEJA VIJAPURAPU

1. Introduction

In this paper we aim to gain an understanding of some of the prerequisites of the Lefschetz Hyperplane Theorem as well as the theorem itself. The theorem, in short, provides a bridge between the shape of algebraic varieties and its subvarieties. We will get into the more rigorous definition near the end of the paper, but it's helpful to note the various connections and uses of this theorem throughout mathematics. More specifically, the Lefschetz Hyperplane Theorem has applications in Hodge Theory, which is a major source of information related to the Hodge conjecture, as well as Morse Theory, which deals with differential topology. Although both of these are well out of the scope of this paper, it is recommended the reader look into them if interested. First, we will need a good understanding of some of the basics of homology and topology, so we begin with that.

2. A crash course in homology and topology

Although homology and topology in themselves are huge branches of mathematics with various applications and perspectives, we will only be dealing with what we need for this paper in understanding the Lefschetz Hyperplane theorem. Roughly speaking, homology groups are a family of topological invariants that don't require complicated computations like some homotopy groups. The goal in homology is to be able to count the number of compact n-dimensional "holes" H in a topological space X. In order to understand some of the concepts in homology further, we will need some basic definitions:

Definition 2.1. We define the *orientation* of an n-simplex S as $S = [v_0, ..., v_n]$ where $v_0, ..., v_n$ are the bounding vertices of S.

The *boundary* of S is

$$\partial_n([v_0, ..., v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, ..., \hat{v}, ..., v_n]$$

where $\hat{v}$ is omitted when summing.

Remark. An 0-simplex is simply a point. A 1-simplex is an edge. A 2-simplex is a triangle, and we generalize this to an arbitrary dimension when we talk about an n-simplex.

Suppose we have a 3-simplex, also known as a tetrahedron. Then, we have the boundary of this as

$$\partial_3([v_0, v_1, v_2, v_3]) = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

The idea of orientations will carry over as the main building block of homology. So, it's helpful to get used to them. Since there is clearly more than one orientation for a given n-simplex —$(n+1)!$ in fact— we need to be able to see whether two orientations are the same or not. In short, we consider two orientations to be equivalent if the permutation connecting them is even, and different if it is odd.

Suppose we have a 2-simplex. The permutation connecting the orientations $[v_0, v_2, v_1]$ and $[v_2, v_0, v_1]$ can be seen as sending $0 \to 2$, $2 \to 0$, and $1 \to 1$, or $(02)(1)$. Since there is 1 transposition, it is an odd permutation, thus the two orientations are different. We call a linear combination of the oriented n-simplices of a space X an n-chain. These n-chains form a group, denoted by $C_n(X)$.

The boundary of an n-simplex forms an $(n-1)$-chain, and from this we can create a homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ defined as $\partial_n(a_1T_1 + \cdots a_1T_r) = a_1\partial_n(T_1) + \cdots a_r\partial_n(T_r)$, where T_i is an n-simplex of a triangulation of a space X and a_i is an element in an n-chain in $C_n(X)$. This homomorphism leads us to the following theorem, which has a helpful corollary.

Theorem 2.2. *Suppose we have an n-chain $A \in C_n(X)$. Then, $\partial_{n-1} \circ \partial_n(A) = 0$.*

Proof. It suffices to just work out the math from definition 2.1.

$$\partial_{n-1} \circ \partial_n(A) = \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i[v_0, ..., \hat{v}_i, ..., v_n]\right)$$

$$= \sum_{i=0}^{n}(-1)^i\partial_{n-1}([v_0, ..., \hat{v}_i, ..., v_n])$$

$$= \sum_{i=0}^{n}(-1)^i\left(\sum_{j=0}^{i-1}(-1)^j[v_0, ..., \hat{v}_j, ..., \hat{v}_i, ..., v_n] + \sum_{j=i+1}^{n}(-1)^{j-1}[v_0, ..., \hat{v}_i, ..., \hat{v}_j, ..., v_n]\right)$$

.

Note that in the last expression, there are two hatted terms instead of the regular one, in different order in both sums. Suppose that $i < j$. Then it appears with coefficient $(-1)^i(-1)^{j-1} = (-1)^{i+j-1}$ (this is when v_i is removed first) and then it appears with coefficient $(-1)^i(-1)^j = (-1)^{i+j}$ (when v_j is removed first). The sum of these is 0, thus for each $(n-2)$-simplex $[v_0, ..., \hat{v}_i, ..., \hat{v}_j, ..., v_n]$ the coefficient is 0, meaning that $\partial_{n-1} \circ \partial_n(A) = 0$. ■

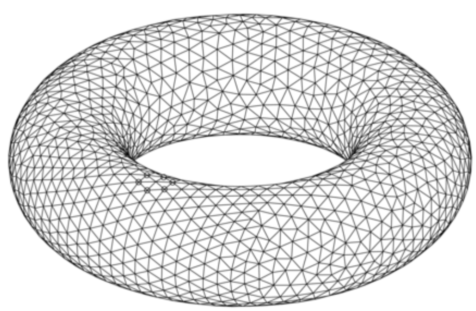
Corollary 2.3. $\text{im}(\partial_n) \leq \ker(\partial_{n-1})$.

Remark. We use "$\leq$" to denote subgroup.

Proof. $\ker(\partial_{n-1})$ contains the $(n-1)$-chains whose boundaries are 0 and $\text{im}(\partial_n)$ contains the $(n-1)$-chains that are images under the boundary map, so it follows that $\text{im}(\partial_n) \leq \ker(\partial_{n-1})$. ■

Definition 2.4. The group of n-cycles $Z_n(X)$ is defined as $\ker(\partial_n) \leq C_n(X)$ and the group of n-boundaries $B_n(X)$ is defined as $\text{im}(\partial_{n+1}) \leq C_n(X)$.

First, we move to an interesting topological concept before formally defining the homology group. We will primarily be focusing on an idea called "triangulation." Simply put, triangulation is the process of dividing a surface into triangles (hence the name triangulation) and doing so such that there is no overlap. Below is an example of a triangulated torus.



We can define this idea more formally as follows:

Definition 2.5. Suppose we have a surface S and a polyhedron P and a homeomorphism $f : P \to S$. f is a *triangulation* if:

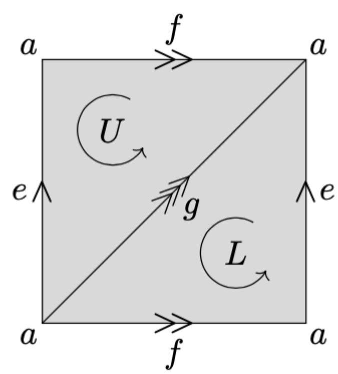
1. Every face of P has three edges and is homeomorphic to a closed disk
2. The intersection of any two faces of P is either empty, a single vertex, or a simple edge
3. The intersection of any two edges of P is either empty or a single vertex

Even though this definition looks heavy at first, it's helpful to always think back to the simpler, less rigorous definition of "splitting" a surface into many, nonoverlapping triangles. You may wonder why this is helpful at all in homology or the final theorem we wish to state, as, at first glance, doesn't seem directly related. However, triangulation is an incredibly helpful tool when it comes to calculating homology groups, which we will see a couple examples of soon, and avoids using complicated techniques. Moving forward with this paper, we will generally think of triangulation in terms of just drawing the edges with the correct properties, rather than constantly deriving and verifying the homeomorphism and its properties listed above.

Now that we have the necessary background for homology groups, let's discuss some of the structure behind them.

Definition 2.6. We call the quotient group $H_n(X) = \frac{Z_n(X)}{B_n(X)}$ the *nth homology group of* X.

The number of holes in X is simply the dimension of $H_n(X)$ (sometimes we say $H_n(X, \mathbb{Z})$ to clarify we are working with coefficients in $\mathbb{Z}$). To get the hang of this, let's do a fairly common examples of calculating the homology group of a torus, then we can move on to relative homology. First, let's use the following diagram for the triangulation of a torus:



We begin by calculating the boundary maps. To find ∂_2, note that U is going "against" f and e, while going "with" g. Thus, $\partial_2(U) = g - f - e$ and using the same logic for L, we get $\partial_2(L) = f + e - g$. These are the 2-chains. To calculate the 1-chains we see that $\partial_1(e) = \partial_1(f) = \partial_1(g) = a - a = 0$. (This is because all these edges start and end at a). Now, let's find $Z_2(\mathbb{T})$ (aka the kernel), where $\mathbb{T}$ is the torus. Clearly, this is just $\langle U + L \rangle$ because $U + L = 0$. We need to find $B_2(\mathbb{T})$ (aka the image) now. This is simply 0 because there does not exist any 3-chains (or you can also say 3-boundaries). Thus, $H_2(\mathbb{T}) = \langle U + L \rangle \cong \mathbb{Z}$. We can move on to $H_1(\mathbb{T})$ now.

We have $Z_1(\mathbb{T}) = \langle e, f, g \rangle$ because these are all 0. $B_1(\mathbb{T}) = \langle e+f-g \rangle$. As you can see, this is very different from $H_2(\mathbb{T})$ because in this quotient group, $g = e + f$. So, we can replace every instance of g with $e + f$. Because of this, every cycle with just e's and f's is considered distinct, thus $H_1(\mathbb{T}) = \langle e, f \rangle \cong \mathbb{Z}^2$. To find $H_0(\mathbb{T})$, we know that $Z_0(\mathbb{T}) = \langle a \rangle$ and $B_0(\mathbb{T}) = 0$. Thus, once again, we have $H_0(\mathbb{T}) \cong \mathbb{Z}$. Now, we have all the calculated components required for the homology group, which is:

$$H_n(\mathbb{T}) \cong \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^2 & n = 1 \\ 0 & n \geq 3 \end{cases}$$

Let's move to relative homology.

3. Relative homology

Now that we have a basic understanding of simplical homology, we can move onto a bit more advanced topic known as relative homology. We will only be covering the basics, as this is all that is needed for our final result.

Before we define the nth relative homology group, we will need a couple more definitions within the scope of simplical homology we skipped over in section 2.

Definition 3.1. A *chain complex* is a sequence of abelian groups connected by the homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ such that $\partial_i \circ \partial_{i+1} = 0$ for $i = 0, ..., n-1$.

From theorem 2.2, we know that any n-chain forms a chain complex. Just like before, the homology groups of a chain complex are of the form $\ker(\partial_n)/\operatorname{im}(\partial_{n+1})$. We also need to define $C_n(X, A)$ (where X is a topological space and $A \subset X$) as $C_n(X)/C_n(A)$. Note that for this to be a valid quotient group, it is necessary that $C_n(A) \leq C_n(X)$. It follows that we can apply the same homomorphism ∂_n to $C_n(X, A)$ and we define it as follows. $\partial_n' : C_n(X, A) \to C_{n-1}(X, A)$.

Definition 3.2. The *nth relative homology group* of a space X and subset $A \subset X$ is the homology group $H_n(X, A)$ of the chain complex of the abelian groups connected by the homomorphism $\partial_n' : C_n(X, A) \to C_{n-1}(X, A)$.

Below is the chain complex we are referring to:

$$\cdots \longrightarrow C_{n+1}(X,A) \xrightarrow{\partial'_{n+1}} C_n(X,A) \xrightarrow{\partial'_n} C_{n-1}(X,A) \longrightarrow \cdots$$

Let's take a look at a more geometric interpretation of the relative homology group. Simplical homology deals with loops that start and end at the same points, but with relative homology, this is not necessarily the case. Both the starting and ending points of the loop must be contained within the space A, but these points need not be the same. It is the homology group of the space X where elements are loops with the condition described above.

4. The lefschetz hyperplane theorem

We now have the necessary background to state and prove the Lefschetz Hyperplane theorem. There are two theorems we will be covering, namely the Lefschetz Main Theorem and the Lefschetz Hyperplane Theorem. Although the proof of the main theorem is far too long to include in this paper, we will be using it to prove the hyperplane theorem. Before we state either theorem, we will need a couple more concepts from topology and homology. The proof of the following theorems are slightly out of the scope of this paper, so we leave it for the reader to look into.

Theorem 4.1 (Five Lemma)**.** *Suppose we have the following commutable diagram:*

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A' & \xrightarrow[r]{} & B' & \xrightarrow[s]{} & C' & \xrightarrow[t]{} & D' & \xrightarrow[u]{} & E' \end{array}$$

If the rows are exact sequences and m and p are isomorphism, l is an epimorphism, and q is a monomorphism, then n is an isomorphism.

Remark. By exact sequence, we mean a sequence of homomorphisms such that the image of one is the kernel of the next.

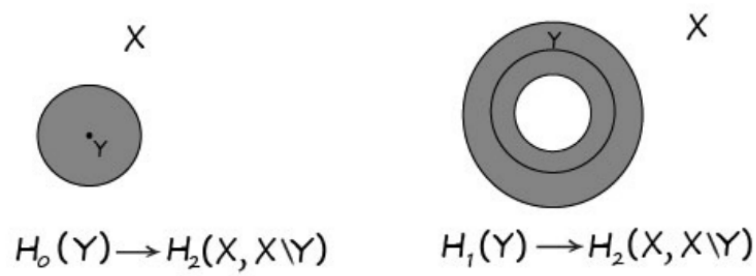
Theorem 4.2 (Leray-Thom-Gysin)**.** *Suppose we have a closed manifold X and a closed oriented submanifold Y of codimension c. Then, the map*

$$H_{m-c}(Y,\mathbb{Z}) \to H_m(X, X\backslash Y, \mathbb{Z})$$

for any m with $H_m(Y) = 0$ if $m < 0$ is an isomorphism.

Remark. By codimension, we mean the complement of the dimension of Y with respect to X.

But what does this map mean? Well, let's say we have a cycle $A \in H_{m-c}(Y)$. What's produced by this map is a closed disk that "grows" transverse to Y in X. The following figure can help to visualize this:



We will be using this isomorphism to help prove the hyperplane theorem. Let's state the main theorem.

Theorem 4.3 (Lefschetz Main)**.** *Suppose we have a smooth projective variety X where Y, Z are two codimension one transversal hyperplane sections of X. Y and Z intersect transversally at X'. Then, we have the following:*

$$H_q(X\backslash Z, Y\backslash X', \mathbb{Z}) = \begin{cases} 0, & \textit{if } q \neq n \\ \textit{free } \mathbb{Z}\textit{-module of finite rank}, & \textit{if } q{=}n \end{cases}$$

Where $n := \dim(X)$.

What this theorem calculated is the homology group of the space $X\backslash Z$, where elements are q-dimensional loops the begin and end at some two points in the space $Y\backslash X'$, however, these points are not necessarily the same.

We can finally state and prove the Lefschetz Hyperplane theorem.

Theorem 4.4 (Lefschetz Hyperplane)**.** *Suppose we have a smooth projective variety X with dimension n. We have smooth hyperplane section $Y \subset X$. Then*

$$H_q(X, Y, \mathbb{Z}) = 0, \ 0 \leq q \leq n-1$$

Proof. Suppose we have the exact sequences:

$$V \subset U \subset X,$$
$$V \subset Y \subset X.$$

The Lefschetz Main Theorem on the first sequence gives us $H_q(X, V) \cong H_q(X, U), \ q \neq n, n+1$ as well as the exact sequence:

$$0 \to H_{n+1}(X,V) \to H_{n+1}(X,U) \to H_n(U,V) \to H_n(X,V) \to H_n(X,U) \to 0$$

We can apply the Leray-Thom-Gysin isomorphism to (U, X) and we get the result $H_q(X,U) \cong H_{q-2}(Z)$. Since $H_q(X,V) \cong X_q(X,U)$, we can conclude that $H_q(X,V) \cong H_{q-2}(Z),\ q \neq n, n+1$.

Let's move to the second exact sequence from our initial two. We can write the sequence for this and $X' \subset Z$ as follows:

$$\begin{array}{ccccccccc} \cdots \to & H_q(Y,V) & \to & H_q(X,V) & \to & H_q(X,Y) & \to & H_{q-1}(Y,V) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H_{q-2}(X') & \to & H_{q-2}(Z) & \to & H_{q-2}(Z,X') & \to & H_{q-3}(X') & \to \cdots . \end{array}$$

The first and fourth arrows here are referring to the Leray-Thom-Gysin isomorphism and the second arrow is the isomorphism $H_q(X,V) \cong H_{q-2}(Z),\ q \neq n, n+1$. The diagram of sequences commutes, so by five lemma, the following holds with regard to the third down arrow:

$$H_q(X,Y) \cong H_{q-2}(Z,X'),\ q \neq n, n+1, n+2.$$

We can use induction on n, thus completing the proof of the Lefschetz Hyperplane Theorem.

■

References

[1] Clark Bray, Adrian Butscher, and Simon Rubinstein-Salzedo. *Algebraic Topology.*

[2] Daniel Halpern-Leistner. Lefschetz hyperplane theorem for stacks. *arXiv preprint arXiv:1008.0891*, 2010.

[3] Hossein Movasati and Roberto Villaflor Loyola. A course in hodge theory. 2019.

[4] Prerna Nadathur. An introduction to homology. *University of Chicago*, 2007.

[5] Wikipedia, the free encyclopedia. Five lemma, 2018. [Online; accessed July 5, 2020].

[6] Wikipedia, the free encyclopedia. Triangulation (topology), 2020. [Online; accessed June 20, 2020].