Klein Quartic Equation

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July 2020

1 Introduction

The Klein Quartic is the curve in $\mathbb{P}^2$ defined by $x^3y + y^3z + z^3x = 0$, where $\mathbb{P}^2$ represents the projective space in 2 dimensions. Or in other words if we take a 3 dimensional space V we can define its projection as $\mathbb{P}(V) = \mathbb{P}^2$. So how can we visualize and sort of construct a reliable image of this klein quartic.

Well lets take regular triangles and glue them s.t 3 triangles meet at a vertice in order to construct a tetrahedron.

Similarly we do the same thing for squares and and regular pentagons to construct cubes and regular dodecahedrons respectively.

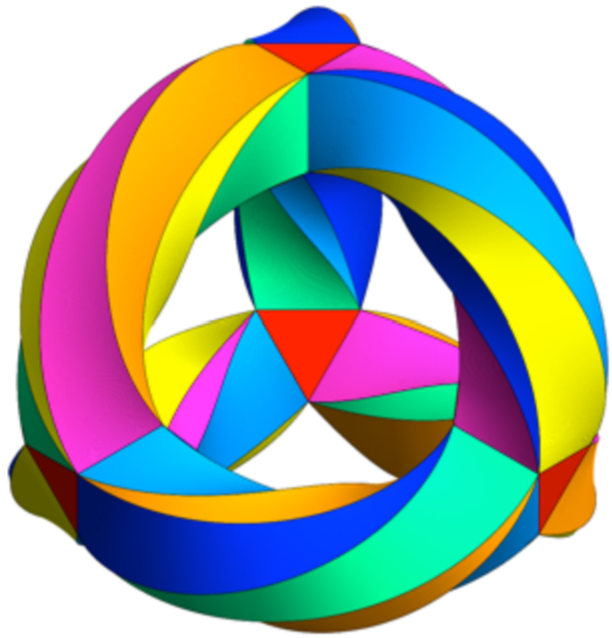
When we try doing this for regular hexagons, however, we cannot construct a typical platonic solid but rather we end up getting a sheet of a honeycomb like pattern which stretches on forever. To get something that is finite, you can take a portion of this sheet and fold it up into a torus or like a donut.

Now we look at regular heptagons. Since the angles of the regular heptagon are greater the 360° when we try to use the same methods used previously we actually get a hyperbolic plane which is kind of the opposite of a sphere. If a sphere bulges outward at every point, the hyperbolic plane bulges inward. This structure is sometimes referred to as 7,3 as it a regular heptagon with 3 meeting at each vertice.

Noe can we do the same thing we did for hexagons and construct some finite structure like a torus. It turns out we can! We can construct a 3 holed torus but only with exactly 24 heptagons. Thus, the klein quartic is born!

The klein quartic has 336 symmetries including rotations and half as many 168 if we don't include them. We also notice that the amount of rotations of heptagon is 7 so $7 * 24 = 168$. If you ever forget this remember the phrase "I work 24-7"!

How exactly do you take the hyperbolic plane and transform it into a klein quartic though. It is only possible to represent the 3 holed torus by warping the hyperbolic plain. To make things easier for ourselves, we first consider 3,7 the structure created by taking regular triangles at having 7 of the meet at a vertice. The 7,3 and the 3,7 overlap very nicely. In fact they are refered to as *Poincare Duals*. If we transform these correctly, we get the 3 holed torus using 56 triangles or 24 heptagons.



This is a visualization of a 3 holed torus.

To further understand what the klein quartic is we have to consider complex analysis. Klein showed that his cool surface could be neatly and symmetrically displayed with a simple equation of 3 complex variables:

$$u^3v + v^3w + w^3v = 0$$

where two solutions are considered the same if they differ by an overall factor.

2 Topology of the KQE

Now as I stated earlier the KQE is based in the complex projective space or $\mathbb{P}^2$ which is a 2 complex dimensional space which contains all rays in $\mathbb{C}^3$. To visualize this, lets create an arbitrary ray in $\mathbb{C}^3$:

$$(xe^{\alpha i}, ye^{\beta i}, z)$$

where $0 \leq \alpha, \beta \leq 2\pi$ and $x, y, z \geq 0$ usch that (x,y,z) is a unit vector in $\mathbb{R}^3$. This tells us that most of $\mathbb{P}^2$ can be thought of as the Cartesian product of the positive octant of a sphere in $\mathbb{R}^3$ (the x,y,z part of the coordinates, where = only two of the coordinates are independent) and a torus (the , coordinates).
To get a clearer picture of the topology, suppose we fix the z coordinate, and allow x and y to vary along an arc of fixed latitude in the positive octant. The collection of tori generated as we move along the arc can be assembled into a 3-sphere. wraps around the constructed cylinder and β ranges from $0 to 2\pi$. As we move along the arc, we go from $x = 0$, $y = \sqrt{(1-z^2)}$, which gives the circle formed by the straight line, to $y = 0$, $x = \sqrt{(1-z^2)}$, which gives the circle.

What we can take away from this is that any arc over a fixed latitude corresponds to a 3 sphere and the size of the 3 sphere is related to z. As z approaches 1 then the whole 3 sphere shrinks to a single point. If z approaches 0 the 3 sphere reaches its maximum size. If we look at this boundary point in $\mathbb{C}^3$ when $z = 0$,

$$(xe^{\alpha i}, ye^{\beta i}, 0)$$

we see that any two points with the same x and y coordinates, whose and coordinates differ by the same amount will be in the same complex ray — the one generated by $(x, ye^{(\beta-\alpha)i}, 0)$. So the set when $z = 0$

turns what we have into a 2-sphere, which basically shifts our dimensions!

If either x or y is also equal to zero, the torus collapses to a single point. So the corners of the octant — (1,0,0), (0,1,0) and (0,0,1) — all represent single points in the whole space $\mathbb{P}^2$ and putting these points back into our original equation:

$$e^{(3\alpha+\beta)i}x^3y + e^{3\beta i}y^3z + e^{\alpha i}z^3x = 0$$

Simplifying this equation by using new coordinates a and b:

$$a = \alpha - 3\beta \tag{2.1}$$

$$b = 3\alpha - 2\beta \tag{2.2}$$

Now dividing by 3β we have

$$e^{ai}x^3y + y^3z + e^{bi}z^3x = 0$$

Now we can find solutions to y/x and z/x

$$y/x = [sin(a–b)^3/(sin(a)^2 sin(b))]^{1/7} \tag{2.3}$$

$$z/x = [sin(b)^2 sin(b–a)/sin(a)^3]^{1/7} \tag{2.4}$$

We said earlier that $x, y, z \geq 0$ so the solutions will only make sense if the seventh roots are both positive. This will occur in fourteen distinct domains in the (α, β) coordinates, separated by lines of the form $a = (2m+1)\pi, b = (2n+1)\pi$, and $a - b = (2k+1)\pi$, where $m, n, k \in \mathbb{Z}$.

Now we can look at the limiting behavior of our functions for our solutions as we approach these borders.

$$\lim_{a\to\ (2m+1)\pi} x/z = \lim_{b\to\ (2m+1)\pi} y/z = 0 \tag{2.5}$$

Basically what this means is when $a = (2m+1)\pi$ we approach the point (0,0,1) and when $b = (2n+1)\pi$ we approach the point (0,1,0) and when $a - b = (2k+1)\pi$ we approach the point (1,0,0).

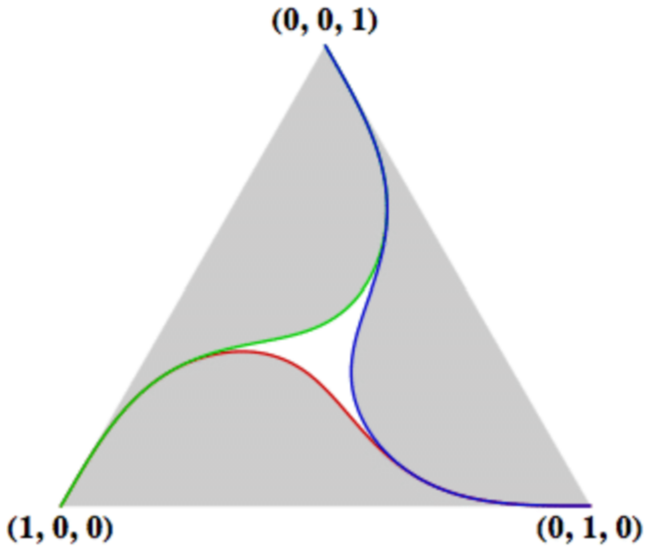
What happens when we approach one of the vertices of the triangular domains in the torus? The map is not continuous on the vertex itself, since the limit jumps abruptly from one point to another point. However, we can take a small line segment that truncates the corner and look at its limiting behaviour as it gets closer to the corner.

If we take the truncating line segment to be $a = \epsilon, b = \pi + \lambda\epsilon$, for $0 \leq \lambda \leq 1$, then take the limit as ϵ goes to zero, we get:

$$\lim_{\epsilon\to\ 0} y/x = \frac{(1-\lambda)^{3/7}}{\lambda^{1/7}} \tag{2.6}$$

$$\lim_{\epsilon\to\ 0} z/x = \frac{(1-\lambda)^{1/7}}{\lambda^{2/7}} \tag{2.7}$$

With these we can solve for x,y,z in terms of λ and we can plot the results. We end up being able to project the octant we have visualized onto an equilateral triangle whose vertices are (1,0,0), (0,1,0), (0,0,1).



For the three curves in the triangle, we have:

red curve: $(a, b) = (0, \pi) \equiv 2\pi \rightarrow -x^3y + y^3z + z^3x = 0$

green curve: $(a, b) = (0, \pi) \equiv 2\pi \rightarrow x^3y + y^3z – z^3x = 0$

blue curve: $(a, b) = (\pi, \pi) \equiv 2\pi \rightarrow x^3y – y^3z + z^3x = 0$

Once we multiply these values of x and y by the appropriate phases $e^{\alpha i}$ and $e^{\beta i}$ we get our solutions to the KQE! These phases come from the vertices of the triangular domains and there are a total of 21 such points from the 14 triangles. If we take the vertices $(\alpha, \beta) = (0, \pi)$, $(\pi, 0)$ and (π, π), they are not shared by a single triangular domain, but their (a, b) values are of each of the three kinds. The phases from these (α, β) values are just ± 1, and taken together, the three curves that project to these vertices comprise the real-valued solutions of KQE.

Hopefully this was a good insight into the properties and topology of the KQE, it was very tough for me to understand!

References

[1] John Baez. *Klein's Quartic Curve*. University of California, Riverside, Riverside, California, 2013.

[2] Greg Egan. *Klein's Quartic Equation*. San Francisco, California, 2006.

[3] Noam D. Elkies. *The Klein Quartic in Number Theory*. Harvard University, Cambridge, Massachusetts, 1998.