Cayley- Hamilton Theorem

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1 Introduction

The Cayley - Hamilton Theorem was first proved in 1853 by mathematicians Arthur Cayley and William Rowan Hamilton. It states that every square matrix over a commutative ring satisfies its own characteristic equation. One of the most common collaries to this theorem is the Jordan Normal Form Theorem which states that any matrix is similar to a block- diagonal matrix with Jordan blocks on the diagonal. This paper assumes basic knowledge of linear algebra, vector spaces, and linear dependence/independence.

2 Cayley - Hamilton Theorem

Let X be a n X n matrix whose values lie over a commutative ring. Let
$f(\lambda) = det(A - \lambda K) = (-1)^n \cdot [\lambda^n + a_1\lambda^{n-1} + ...a_n]$
be the characteristic polynomial of A. Than the following holds true:

$$f(X) = (-1)^n \cdot [X^n + a_1 A^{n-1} ... + a_n K] = 0$$

3 Proof of Cayley - Hamilton Theorem

We have the following:

$$(A - \lambda K) \cdot adj(A - \lambda K) = det(A - \lambda K) = f(x) \cdot K$$

and we get that the adjoin matrix has the form:

$$adj(A - \lambda K) = (f_(ij)(\lambda))$$

where $f_(ij)(\lambda)$ are polynomials in λ of degree at most $n-1$ for $1 \leq i, j \leq n$.
We can than rewrite the adjoint matrix as

$$adj(A - \lambda K) = B_0 + B_1\lambda + ... + B_(n-1)\lambda^{n-1}$$

for some $nbyn$ matrices $B_0, B_1..B_(n-1)$. From here we can equation coefficients to get the following:

$$AB_0 = (-1)^n a_n K \; -B_0 + AB_1 = (-1)^n a_n - 1K \; . \; . \; .$$
$$-B_(n-2) + AB_(n-1) = (-1)^n a_1 K \; -B_(n-1) = (-1)^n K.$$

From here we can multiply each of these equations respectively with $I, A, ...A^{n-1}, A^n$ to get the following:

$$\begin{aligned} AB_0 &= (-1)^n a_n K \\ -AB_0 + A^2B_1 &= (-1)^n a_{n-1} K \; . \; . \; . \\ -A^{n-1}B_{n-2} + AB_{n-1} &= (-1)^n a_1 A^{n-1} \\ -A^n B_{n-1} &= (-1)^n A^{n-1}. \end{aligned}$$

From here we can proceed and see the sum on the left telescopes to the zero matrix while the sum on the right is just f(A). Therefore we have achieved our desired result.

4 Daily Applications of Cayley - Hamilton Theorem

The Cayley - Hamilton is often expressed in many of the products arounds; something we often fail to recognize. Notable fields which use this theorem for designing their products are automation, quantum mechanics, electrical engineering, etc. These fields all require the applications the of matrices and the Cayler - Hamilton theorem helps simplify the matrix into a polynomial equation. In quantum mechanics, the Cayley - Hamilton Theorem assists with finding characteristic roots of a given equation. Additionally, in electircal engineering the Cayley - Hamilton theorem is used for converting matrices to polynomial equations that the control board will be able to process and manipulate in order to achieve the desired result.

5 Cayley - Hamilton's Applications in 3 - Dimensions

The applications of Cayley - Hamilton in 3 Dimensions is one which not much recognized amongst individuals. This theorem is very well known in this branch of mathematics as one can split the 3 dimensions simply into 3

2 by 2 matrices, hence simplifying the computational process a lot. These matrices in return can than be written as a polynomial. Within a matter of 2 steps, we have converted a sophisticated batch of coordinates into easily usable polynomials which can be manipulated to produce more results.

References

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