Homological Algebra

Merrick Hua

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1 Basic Category Theory

In order to study homological algebra, it is essential that one is familiar with the language of category theory and basic categorical concepts. In this section, we briefly define categories and functors.

1.1 Categories

Loosely speaking, a category is a collection of objects and maps between these objects, which we can compose uniquely to get new maps. Furthermore, the composition of these maps is associative, and for each object in the category there exists an identity map from it to itself that does nothing when composed with other maps. More formally, we have the following definition:

Definition 1.1.1 (Categories)**.** A **category** $\mathscr{C}$ consists of the following:

- A collection of objects, denoted $\mathrm{Ob}(\mathscr{C})$.
- For any $A, B \in \mathrm{Ob}(\mathscr{C})$, a collection of morphisms from A to B, denoted $\mathrm{Mor}_{\mathscr{C}}(A, B)$.
- For any $A, B, C \in \mathrm{Ob}(\mathscr{C})$, $f \in \mathrm{Mor}_{\mathscr{C}}(A, B)$, and $g \in \mathrm{Mor}_{\mathscr{C}}(B, C)$, an unique $g \circ f \in \mathrm{Mor}_{\mathscr{C}}(A, C)$, which we call the composition of f and g.

These objects and morphisms have to satisfy a few properties:

1. For any $A \in \mathrm{Ob}(\mathscr{C})$, there exists an identity morphism $\mathrm{id}_A \in \mathrm{Mor}_{\mathscr{C}}(A, A)$ such that for any $B \in \mathrm{Ob}(\mathscr{C})$, $f \circ \mathrm{id}_A = f$ for any $f \in \mathrm{Mor}_{\mathscr{C}}(A, B)$, and $\mathrm{id}_A \circ g = g$ for any $g \in \mathrm{Mor}_{\mathscr{C}}(B, A)$.
2. For $f \in \mathrm{Mor}_{\mathscr{C}}(A, B)$, $g \in \mathrm{Mor}_{\mathscr{C}}(B, C)$, and $h \in \mathrm{Mor}_{\mathscr{C}}(C, D)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$, i.e., composition of morphisms is associative.

Some things to note:

- Morphisms are also sometimes referred to as maps or arrows. It is important to recognize that morphisms do not have to be functions between sets (hence the term "arrow"), although the vast majority of categories we will work with in this paper will have functions as morphisms.
- When the context is clear, we will simply write $\mathrm{Mor}(A, B)$ in place of $\mathrm{Mor}_{\mathscr{C}}(A, B)$.
- We often will write $A \in \mathscr{C}$ instead of $A \in \mathrm{Ob}(\mathscr{C})$.
- Rather than writing $f \in \mathrm{Mor}(A, B)$, we will often write $f\colon A \to B$, or $A \xrightarrow{f} B$. We say that A is the domain of f, and B is the codomain.
- Neither $\mathrm{Ob}(\mathscr{C})$ or $\mathrm{Mor}(A, B)$ for any $A, B \in \mathscr{C}$ have to be proper sets; this is why we say collection as opposed to set.

1.2 Functors

Structure preserving maps are incredibly important in mathematics. Naturally, we wish to study structure preserving maps between categories, which are called functors. Functors come in two flavors, covariant and contravariant, both of which are introduced below.

Definition 1.2.1 (Covariant Functors)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **covariant functor** $F\colon \mathscr{C} \to \mathscr{D}$ consists of the following:

- A map from $\mathrm{Ob}(\mathscr{C})$ to $\mathrm{Ob}(\mathscr{D})$, such that if $A \in \mathrm{Ob}(\mathscr{C})$, then $F(A) \in \mathrm{Ob}(\mathscr{D})$.
- For any $A, B \in \mathscr{C}$, a map from $\mathrm{Mor}_{\mathscr{C}}(A, B)$ to $\mathrm{Mor}_{\mathscr{D}}(F(A), F(B))$, such that if $f \in \mathrm{Mor}_{\mathscr{C}}(A, B)$, then $F(f) \in \mathrm{Mor}_{\mathscr{D}}(F(A), F(B))$.

  These maps have to satisfy a few conditions:

  1. Let A be any object in $\mathscr{C}$, and let id_A be the identity morphism associated with A. Then, we must have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.
  2. If $f\colon A \to B$ and $g\colon B \to C$ are morphisms in $\mathscr{C}$, then $F(g \circ f) = F(g) \circ F(f)$.

To summarize, covariant functors send objects and morphisms from one category to objects and morphisms in another category, in a way that preserves identity morphisms and composition of morphisms.

Similarly, we have contravariant functors, which reverse the order of composition instead.

Definition 1.2.2 (Contravariant Functors)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **contravariant functor** $F\colon \mathscr{C} \to \mathscr{D}$ satisfies all the conditions for a covariant functor, except that condition 2 is changed as follows:

If $f\colon A \to B$ and $g\colon B \to C$ are morphisms in $\mathscr{C}$, then $F(g \circ f) = F(f) \circ F(g)$.

For more detailed material on categories and functors, consider reading [Lei16] or [FS18].

2 Modules and Hom Functors

Modules are a generalization of the concept of a vector space over a field: modules are over a ring instead. Modules are a central component in the study homological techniques. In this section, we will define what a module is, explore categories of modules, and examine the Hom functor.

2.1 Modules

Definition 2.1.1 (Modules)**.** Given a commutative ring with identity R, a **left R-module** M is an abelian group $(M, +)$ along with a map $\cdot\colon R \times M \to M$ (which sends (r, m) to $r \cdot m$) satisfying the following properties:

1. $(r + s) \cdot m = r \cdot +s \cdot m$ for all $r, s \in R$, $m \in M$.
2. $(rs) \cdot m = r \cdot (s \cdot m)$ for all $r, s \in R$, $m \in M$.
3. $r \cdot (m + n) = r \cdot m + r \cdot n$ for all $r \in R$, $m, n \in M$.
4. $1_R \cdot m = m$ for all $m \in M$.

In order to be a module, we also need a map $\star : M \times R \to M$ sending (m, r) to $m \star r$, and we require $r \cdot m = m \star r$. So, we have versions of the above four properties for $\star$, except the elements of R are written on the right instead.

Remark: It's definitely easier to remember this definition by thinking of a module as a generalized vector space: instead of the scalars coming from a field, they come from a ring.

The reason we have the issue of writing elements of R on the left or right is because we want to be able to define modules over noncommutative rings as well. However, in this paper, we will only work with commutative rings, so it is fine to assume that elements of R can always be written on the left. For sake of notation, we will often write rm instead of $r \cdot m$.

Let's look at a few examples of modules:

Example 2.1.2. Any commutative ring R is a R-module; that is, any ring is a module over itself. It is clear that if we let $\cdot\colon R \times R \to R$ be the ring product, all the module conditions are satisfied.

Example 2.1.3. Every abelian group is a $\mathbb{Z}$-module. If G is an abelian group, we can have $\mathbb{Z}$ act on G as follows:

For $n \in \mathbb{Z}$, $g \in G$,

$$ng = \begin{cases} g + g + \cdots + g \text{ (n times)} & \text{if } n > 0 \\ 0 & \text{if } n > 0 \\ -g - g - \cdots - g \text{ (n times)} & \text{if } n < 0 \end{cases} \quad (1)$$

Example F. or a concrete example of how any abelian group is a $\mathbb{Z}$-module, consider $\mathbb{Z}/2\mathbb{Z}$. Using the action of $\mathbb{Z}$ defined above, we have

$$\begin{cases} n \cdot 1 = 0 & \text{if } n \equiv 0 \pmod 2 \\ n \cdot 1 = 1 & \text{if } n \equiv 1 \pmod 2 \\ n \cdot 0 = 0 \end{cases} \quad (2)$$

It is not difficult to see that this makes $\mathbb{Z}/2\mathbb{Z}$ a $\mathbb{Z}$-module (verify this for yourself!)

As is the case with any interesting algebraic structure, we want to consider structure-preserving maps between modules, which we call module homomorphisms.

Definition 2.1.5 (Module Homomorphisms)**.** Let R be a ring, and M, N be R-modules. We say a map $\phi\colon M \to N$ is a **R-module homomorphism** if it satisfies the following conditions:

1. $\phi(x + y) = \phi(x) + \phi(y)$, for all $x, y \in M$.
2. $\phi(rx) = r\phi(x)$, for all $r \in R$, $x \in M$.

This definition is very similar to that of a ring homomorphism; module isomorphisms and kernels of module homomorphisms are analogously defined.

Definition 2.1.6. Let R be a ring, and M, N be R-modules. We define $\mathrm{Hom}_R(M, N)$ to be the set of all R-module homomorphisms from M to N. When the context is clear, we simply write $\mathrm{Hom}(M, N)$.

We can then define a category of R- modules for a given ring R:

Definition 2.1.7 (The Category of R-Modules)**.** Let R be a ring. The category of R-modules, R-**Mod**, has R-modules as objects, and R-module homomorphisms as morphisms. Specifically, if M, N are R-modules, then $\text{Mor}(M, N) = \text{Hom}_R(M, N)$.

You should verify for yourself that this is indeed a category (composition of R-module homomorphisms are also R-modules, composition is associative, there exists identity R-module homomorphisms, etc.). This category is very important, and will be the setting for most of the math in the rest of this paper.

2.2 Hom Functors

Let R be a ring, and M, N be an R-modules. By introducing an addition of homomorphisms, we can make $\text{Hom}_R(M, N)$ into an abelian group. Suppose that $f, g \colon M \to N$ are R-module homomorphisms. Then, we can define a map $f + g \colon M \to N$ such that $(f + g)(x) = f(x) + g(x)$ for all $x \in M$, which is easily seen to be a R-module homomorphism. With this addition operation, $\text{Hom}_R(M, N)$ becomes an abelian group as desired.

We now define two very important functors from R-**Mod** to **Ab** (the category of abelian groups, which has abelian groups as objects and group homomorphisms as morphisms).

Definition 2.2.1 (Covariant Hom Functor)**.** Let R be a ring and D be a R-module. Then, $\text{Hom}(D, -) \colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is a covariant functor which does the following:

- $\text{Hom}(D, -)$ maps each R-module $N \in R$-**Mod** to the abelian group $\text{Hom}(D, N)$.
- $\text{Hom}(D, -)$ maps each R-module homomorphism $\phi \colon N_1 \to N_2$ to the group homomorphism $\text{Hom}(D, \phi) \colon \text{Hom}(D, N_1) \to \text{Hom}(D, N_2)$, such that $\text{Hom}(D, \phi)$ sends $\psi \in \text{Hom}(D, N_1)$ to $\phi \circ \psi \in \text{Hom}(D, N_2)$.

Let us check that this is actually a functor; that R-module homomorphisms are indeed sent to group homomorphisms, and that identities are preserved. Let $\phi \colon N_1 \to N_2$ be a R-module homomorphism, and let ψ_1, ψ_2 be in $\text{Hom}(D, N_1)$. We want to show that $\text{Hom}(D, \phi)$ is a group homomorphism; in other words, we wish to show that $\phi \circ (\psi_1 + \psi_2) = \phi \circ \psi_1 + \phi \circ \psi_2$. However, for any $x \in D$, we have $\psi_1(x), \psi_2(x) \in N_1$. Since ϕ is a R-module homomorphism, we then must have $\phi(\psi_1(x) + \psi_2(x)) = \phi(\psi_1(x)) + \phi(\psi_2(x))$. This then shows that $\phi \circ (\psi_1 + \psi_2) = \phi \circ \psi_1 + \phi \circ \psi_2$, as desired. Furthermore, if we consider the identity homomorphism $\text{id}_{N_1} : N_1 \to N_1$, then $\text{Hom}(D, \text{id}_{N_1})$ sends $\phi \in \text{Hom}(D, N_1)$ to $\text{id}_{N_1} \circ \phi \in \text{Hom}(D, N_1)$, so $\text{Hom}(D, \text{id}_{N_1})$ is clearly the identity morphism for $\text{Hom}(D, N_1)$. Thus, $\text{Hom}(D, -)$ preserves identity morphisms, as desired.

There also exists a very similar contravariant Hom functor:

Definition 2.2.2 (Contravariant Hom Functor)**.** Let R be a ring and D be a R-module. Then, $\mathrm{Hom}(-, D)\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is a contravariant functor which does the following:

- $\mathrm{Hom}(-, D)$ maps each R-module $N \in R\text{-}\mathbf{Mod}$ to the abelian group $\mathrm{Hom}(N, D)$.
- $\mathrm{Hom}(-, D)$ maps each R-module homomorphism $\phi\colon N_1 \to N_2$ to the group homomorphism $\mathrm{Hom}(\phi, D)\colon \mathrm{Hom}(N_2, D) \to \mathrm{Hom}(N_1, D)$, such that $\mathrm{Hom}(\phi, D)$ sends $\psi \in \mathrm{Hom}(N_2, D)$ to $\psi \circ \phi \in \mathrm{Hom}(N_1, D)$.

The verification that $\mathrm{Hom}(-, D)$ is a functor is basically the same as the verification that $\mathrm{Hom}(D, -)$ is a functor, so we omit it.

These two functors are actually specialized cases of more general functors: you can define a covariant Hom functor and a contravariant Hom functor from any (locally small*) category to **Set**. However, since our primary focus is on modules, we will not delve further into this.

*Locally small means that for any objects A, B in the category, $\mathrm{Mor}(A, B)$ is a proper set.

3 Exact Sequences

Homological algebra derives its name from homology, a notion in algebraic topology. So, it comes as no surprise that exact sequences, which play a large role in homology, are also a key concept in homological algebra. In this section, we define short and long exact sequences, and look at functors which preserve exactness.

3.1 Short and Long Exact Sequences

We first define what it means for a sequence of modules and module homomorphisms to be exact.

Definition 3.1.1 (Exact Sequences)**.** Let R be a ring, $M_1, \ldots, M_n$ be R-modules, and $\phi_1, \ldots, \phi_{n-1}$ be maps such that $\phi_i\colon M_i \to M_{i+1}$ is a R-module homomorphism for all $1 \leq i \leq n-1$. We say that the sequence

$$M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} M_n$$

is **exact at** $\mathbf{M_i}$ if we have $\mathrm{im}(f_i) = \ker(f_{i+1})$. If the sequence is exact at M_i for all $1 \leq i \leq n-1$, then we say that the sequence is an **exact sequence**.

Let's take a look at some almost-trivial examples of exact sequences:

Theorem 3.1.2. Let R be a ring, and M, N be R-modules. We have the following results:

1. The sequence $0 \to M \xrightarrow{\phi} N$ is exact if and only if ϕ is injective.

2. The sequence $M \xrightarrow{\phi} N \to 0$ is exact if and only if ϕ is surjective.

3. The sequence $0 \to M \xrightarrow{\phi} N \to 0$ is exact if and only if M and N are isomorphic.

Proof.

1. Suppose that the sequence is exact. The unique homomorphism from 0 to M must send 0 to 0, so the image of that homomorphism is 0, which in turn shows that the kernel of ϕ is 0 as well (since the sequence is exact). However, ϕ is injective precisely when $\ker(\phi)$ is 0. For the other direction, if ϕ is injective, then $\ker(\phi)$ is 0, and it is easy to see that the sequence is exact.

2. Suppose that the sequence is exact: the unique homomorphism from N to 0 must send all of N to 0, so the kernel of that homomorphism is all of N. Then, the exactness of the sequence implies that $\text{im}(\phi) = N$, which clearly implies that ϕ is surjective. For the other direction if ϕ is surjective, then $\text{im}(\phi) = N$, and it is easy to see that the sequence is exact.

3. Suppose that the sequence is exact. That implies that the subsequences $0 \to M \xrightarrow{\phi} N$ and $M \xrightarrow{\phi} N \to 0$ are exact as well. As shown in part 1. and 2., ϕ must be injective and surjective, and thus a isomorphism. For the other direction, if ϕ is an isomorphism, then splitting the sequence into the two subsequences and regarding ϕ as injective for the first and surjective for the second reduces this proof to the previous two parts.

□

Part 3 of Theorem 3.1.2 shows that exact sequences with not many nonzero modules involved are not very interesting. We examine the smallest case where things do become interesting:

Definition 3.1.3 (Short Exact Sequences)**.** If R is a ring and M, N, L are R-modules, and the sequence $0 \to M \xrightarrow{\phi} N \xrightarrow{\psi} L \to 0$ is exact, we call it a **short exact sequence**.

We also have long exact sequences:

Definition 3.1.4 (Long Exact Sequences)**.** If R is a ring and $M_1, \ldots, M_n$ are R-modules, and the sequence $0 \to M_1 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{n-1}} M_n \to 0$ is exact with $n > 3$, then we call the sequence a **long exact sequence**.

Example 3.1.5. The sequence $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/2\mathbb{Z} \to 0$ is an exact sequence, where ϕ is the homomorphism that sends $a \in \mathbb{Z}/2\mathbb{Z}$ to $(a, 0) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and ψ is the homomorphism that sends $(a, b) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ to $b \in \mathbb{Z}/2\mathbb{Z}$.

Note that in the above example, ϕ and ψ are injective and surjective, respectively, so Theorem 3.1.2 implies that $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/2\mathbb{Z} \to 0$ are both exact. Furthermore, $\text{im}(\phi) = \{(a, 0) \mid a \in \mathbb{Z}/2\mathbb{Z}\} = \ker(\psi)$, so the entire sequence is exact.

3.2 Left and Right Exact Functors

Since an exact sequence of R-modules consists of R-modules and R-module homomorphisms, it is tempting to apply a functor to the sequences and see what happens. Hopefully, the functor will preserve exactness, which leads to the following definition:

Definition 3.2.1 (Left and Right Exact Functors)**.** Let F be a covariant functor from R-**Mod** to some category in which we can speak about exact sequences (for example, we could have the category be **Ab**). Let $0 \to M \xrightarrow{\phi} N \xrightarrow{\psi} L \to 0$ be a short exact sequence of R-modules. If $0 \to F(M) \xrightarrow{F(\phi)} F(N) \xrightarrow{F(\psi)} F(L)$ is exact, we say that F is **left exact**. Similarly, if $F(M) \xrightarrow{F(\phi)} F(N) \xrightarrow{F(\psi)} F(L) \to 0$ is exact, we say that F is **right exact**. If F is both left and right exact, then it is an **exact functor**.

Analogously, let G be a contravariant functor from R-**Mod** to some suitable category, and let $0 \to M \xrightarrow{\phi} N \xrightarrow{\psi} L \to 0$ be a short exact sequence of R-modules. If $0 \to G(L) \xrightarrow{G(\psi)} G(N) \xrightarrow{G(\phi)} G(M)$ is exact, then G is left exact. If $G(L) \xrightarrow{G(\psi)} G(N) \xrightarrow{G(\phi)} G(M) \to 0$ is exact, then G is right exact.

It turns out we have already seen two left exact functors: $\text{Hom}(D, -)$ and $\text{Hom}(-, D)$.

Theorem 3.2.2. The functors $\text{Hom}(D, -)$ and $\text{Hom}(-, D)$ are left exact, where D is a R-module.

Proof. Check Chapter 10 of [DF04] for a proof. □

4 Derived Functors: Ext

It is rather unsatisfying that $\mathrm{Hom}(D, -)$ and $\mathrm{Hom}(-, D)$ are left exact, and not always exact. In this section, we introduce a functor which quantifies this failure to be exact.

4.1 Projective and Injective Resolutions

Definition 4.1.1 (Projective Modules)**.** We say a R-module P is projective if the following holds:

For R-modules L, M, N, if $0 \xrightarrow{L} \xrightarrow{\phi} M \xrightarrow{\psi} N \xrightarrow{0}$ is a short exact sequence, then the sequence $0 \to \mathrm{Hom}(P, L) \xrightarrow{\mathrm{Hom}(P,\phi)} \mathrm{Hom}(P, M) \xrightarrow{\mathrm{Hom}(P,\psi)} \mathrm{Hom}(P, N)$ is also a short exact sequence.

Projective modules are those such that the functor $\mathrm{Hom}(P, -)$ is not just left exact, but exact. There is a similar definition for modules which make the contravariant Hom functor exact:

Definition 4.1.2 (Injective Modules)**.** We say a R-module Q is injective if the following holds:

For R-modules L, M, N, if $0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \xrightarrow{0}$ is a short exact sequence, then the sequence $0 \to \mathrm{Hom}(Q, N) \xrightarrow{\mathrm{Hom}(P,\phi)} \mathrm{Hom}(Q, M) \xrightarrow{\mathrm{Hom}(P,\psi)} \mathrm{Hom}(Q, L)$ is also a short exact sequence.

Now, we can define projective resolutions:

Definition 4.1.3 (Projective Resolutions)**.** Let M be a R-module, and let $P_0, P_1, \ldots$ be projective R-modules. If $\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$ is an exact sequence, we call the sequence a projective resolution of M.

We also have a similar notion of injective resolutions, which we will not write out here (an injective resolution is basically a sequence of maps out of zero, which extend M using injective modules).

4.2 The Ext Functor

Definition 4.2.1 (The Contravariant Ext Functors)**.** Let M and D be R-modules. For any projective resolution of M, let d_n be the map from $\mathrm{Hom}_R(P_{n-1}, D)$ to $\mathrm{Hom}_R(P_n, D)$. We let $\mathrm{Ext}^n_R(M, D) = \ker(d_{n+1}) / \operatorname{im}(d_n)$; the functors $\mathrm{Ext}^n_R(-, D)$ are called the derived functors of $\mathrm{Hom}_R(-, D)$.

We also have a similar definition for the covariant Ext Functors, which are derived from $\mathrm{Hom}_R(D, -)$

The reason these functors are called Ext is that they extend the left exact sequences obtained by applying the Hom functor:

Theorem 4.2.2. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Then, the following is a long exact sequence:

$$0 \to \mathrm{Hom}_R(D, L) \to \mathrm{Hom}_R(D, M) \to \mathrm{Hom}_R(D, N) \xrightarrow{\gamma_0} \mathrm{Ext}^1_R(D, L)$$
$$\to \mathrm{Ext}^1_R(D, M) \to \mathrm{Ext}^1_R(D, N) \xrightarrow{\gamma_1} \mathrm{Ext}^2_R(D, L) \to \cdots$$

So, the Ext functors extend these exact sequences.

References

[DF04] David Steven Dummit and Richard M Foote. *Abstract Algebra*, volume 3. Wiley Hoboken, 2004.

[FS18] Brendan Fong and David I Spivak. Seven sketches in compositionality: An invitation to applied category cheory. 2018. `https://arxiv.org/abs/1803.05316`.

[Lei16] Tom Leinster. Basic category theory. 2016. `https://arxiv.org/pdf/1612.09375.pdf`.