EULER CIRCLE PAPER: HOMOLOGICAL ALGEBRA

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Abstract. In this paper, I hope to cover some of the important and interesting topics in Homological Algebra, such as algebraic topology (homeomorphic topological spaces, specifically, homomorphisms between nth homology groups of topological spaces for all n.), ring structures, and more. I will be citing from [GQ16] and [Nad07][1].

1. Introduction to Homology: Definitions and Basics

We begin by stating some basic definitions which will be useful later on.

Definition 1.1. We call the geometric figure which can be created from a collection of $n+1$ points in $\mathbb{R}^n$ the *n-simplex*[2]. Here are some basic, very simple examples: point, segment, triangular prism, rectangular prism, pentagonal prism, hexagonal prism, etc.

As we can see, each n-simplex is either just a point, segment, or regular prism.

Definition 1.2. An *n-face* of an n-simplex is a subset of the set of vertices of the simplex with order $n+1$. We call the faces with dimension less than n *proper faces*.

Let's introduce the notion of a *complex*:

Definition 1.3. A *complex* is a finite set of vertices $\{v_0, \ldots, v_k\}$ where some subsets are abstract simplices, and all faces of distinguished simplices are distinguished. Equivalently, we could define the complex (also called the abstract simplicial complex) as a family F of subsets

Date: July 13, 2020.

[2]Note that the plural of simplex is simplices, which we'll see often.

of a set S where, for every set $S' \in F$ and subset $R \subseteq S'$, we have that $R \in F$.

A *subcomplex* is a subset S of the simplices of a complex K such that S is also complex.

A complex K is said to be *connected* if it cannot be represented as the disjoint union of at least two subcomplices. A geometric complex is said to be *path-connected* if there is some path of 1-simplices from a vertex to another.

In addition to a complex, we also have a *geometric realization*, which is essentially a function taking elements of the set $\{v_0, v_1, v_2, \ldots, v_k\}$ (see the latter definition) and mapping them to points in $\mathbb{R}^n$.[3] Now let's look at a definition combining some things we've just learnt:

Definition 1.4. A *simplicial complex* K is a finite set of simplices which satisfies the following conditions:

(1) We have $a \in K$ for all simplicies $A \in K$ where a is a face of A.
(2) We have that $A, B \in K$ are properly situated.[4]

Now let's turn to some definitions more directly focused on homology groups.

Definition 1.5. Let S be the set of vertices of a simplex. Selecting one ordering of the elements of S gives us an *orientation.* We call an odd permutation *reversed* and an even permutation *unchanged.*

The next few definitions will be rather dense and a bit hard to unpack, but they will provide a basis for us to work off of.

Definition 1.6. Given a set $A_1^n, A_2^n, \ldots, A_k^n$ of oriented n-simplices (with complex K and some abelian group G and $g_i \in G$), the *n-chain* x with coefficients in G is the following formal sum:

$$x = g_1 A_1^n + g_2 A_2^n + \ldots + g_k A_K^n.$$

We denote the group of n-chains as L_n.

Remark 1.7. While we could certainly allow G to be some arbitrary abelian group, for now let's just assume G is $\mathbb{Z}$.

Next, let's introduce *boundaries.*

[3] One such example is the *natural realization*, where we let $n = k + 1$ and $v_0 = e_1, v_1 = e_2, \ldots, v_k = e_{k+1}$. Here, the $e_i \in \mathbb{R}^n$ are basis vectors.

[4] We say that two simplices are *properly situated* if their intersection is either empty, or it is a simplex itself.

Definition 1.8. Suppose A^n is an oriented n-simplex in some complex K. The $(n-1)$-chain of K over $\mathbb{Z}$ from

$$\delta(A^n) = A_0^{n-1} + A_1^{n-1} + \ldots + A_n^{n-1}$$

(here, A_i^{n-1} is an $(n-1)$-face of K over $\mathbb{Z}$) is called the *boundary* of A^n.[5] If an n-chain has boundary 0, we call the n-chain a *cycle*, and write the set of n-cycles of K over $\mathbb{Z}$ as $\text{Ker}(\delta) = Z_n$.

A more general (for all L_n), more complicated definition of the boundary is as follows: Suppose we have some n-chain $x = \sum_{i=1}^{k} g_i A_i^n$. Then we have

$$\delta(x) = \sum_{i=1}^{k} g_i \delta(A_i^n),$$

where the A_i^n are n-simplices of K. So we have that the boundary function δ is the homomorphism $\delta : L_n \to L_{n-1}$.

Now that we have seen the basics, we can focus more on actual homology.

2. The Homology group

We will begin our study of the homology group with the following definition, which relies on several of the previous definitions.

Definition 2.1. An n-cycle x of a k-complex K is said to be *homologous to zero* if it is the boundary of an $(n+1)$-chain of K, for all $n = 0, 1, 2, \ldots, k-1$. We write this as $x \sim 0$. The subgroup of Z_n of boundaries is denoted as B_n.

From this, we can now say that a boundary is any cycle homologous to zero. Now we can finally define the *homology group*:

Definition 2.2. The n-dimensional *homology group* of the complex K over $\mathbb{Z}$ is the group $H_n = Z_n/B_n$.

In the latter definition, we have that B_n is a subgroup of Z_n; this is why we can form the quotient group Z_n/B_n. Now that we've seen all these definitions, it's time for a theorem:

Theorem 2.3. *Let $\{K_1, K_2, \ldots, K_p\}$ be the set of all connected components of a complex K, and let H_n, $H_n i$ be the homology groups of K and K_i, respectively. Then we have that H_n is isomorphic to the direct sum $H_{n1} \oplus \cdots \oplus H_{np}$.*

[5] If $n = 0$, we have that $\delta(A^0) = 0$.

Proof. Let L_{ni} be the group of n-chains of K_i.[6] Then L_{ni} must be a subgroup of L_n. Additionally, we have that $L_n = L_{n1} \oplus \cdots \oplus L_{np}$. We need to show that something similar is true for B_n and Z_n.

Let $B_{ni} = \delta(L_{n+1i}) = \mathrm{Im}(\delta)$, where this image is restricted to the subgroup L_{ni}. We can then write B_n as the following direct sum:

$$B_n = B_{n1} \oplus \cdots \oplus B_{np}.$$

Consider some element $x \in L_{n+1}$, where $x = x_i + \ldots + x_p$. Now, for $x_i \in L_{n+1i}$, we have the following:

$$\delta x = \delta x_1 + \ldots + \delta x_p \in B_n.$$

Let $Z_{ni} = \mathrm{Ker}(\delta) \cap L_{ni}$. We have

$$Z_n = Z_{n1} \oplus \cdots \oplus Z_{np}.$$

It is not too difficult to show this is true.

Now that we've shown that both B_{ni} and Zni break down componentwise, we have that

$$Z_n/B_n = Z_{n1}/B_{n1} \oplus \cdots \oplus Z_{np}/B_{np}$$

and also that

$$H_n = H_{n1} \oplus \cdots \oplus H_n p.$$

This completes the proof. [Nad07] □

Definition 2.4. The *index* of a chain $x = \sum_{i=1}^{k} g_i A_{ni}$ is

$$I(x) = \sum_{i=1}^{k} g_i.$$

Proposition 2.5. *If K is a connected complex, then for some 0-chain x, we have that $I(x) = 0 \implies x \sim 0$, and also that $H_0(K, \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. See Nadathur's proof. □

Now let's look at a lemma, which will help us with the proof of the next theorem.

Lemma 2.6. *A geometric complex K is path-connected iff it is connected.*

[6] Recall that L_n is the group of n-chains of K, and that K_i represents the i^{th} component of K.

Proof. Let's first prove that a geometric complex K is path-connected if it is connected. Suppose that K is not connected. Then there exist disjoint subcomplexes L and M where $L \cup M = K$. Furthermore, suppose there is some path between a vertex $v_0 \in V$ and $n_0 \in N$. If v_i the last vertex in the path contained in V, we don't have that the 1-simplex connecting v_i to the next vertex in the path is contained in V or N. This is because we would have $V \cap N = \varnothing$, which contradicts our initial assumption.

Next, we prove that K is connected if it is path-connected. Assume we have points $v_0, n_0 \in K$ such that there is no path between them. Then V is the path-connected subcomplex of K containing v_0, and similarly N is the path-connected subcomplex of K containing n_0. There must be some path connecting v_0 to t_0 and a path connecting t_0 to n_0 for some t_0 in the non-empty intersection $V \cap N$. Because there are two paths, one ending with t_0 and the other beginning with t_0, we can simply create the path v_0 to n_0, but this contradicts our assumption (that there is a path between these two). Therefore, $V \cap N = \varnothing$. □

Now we can use the previous lemma and Definition 1.12 for the following theorem:

Theorem 2.7. *The zero-dimensional homology group of a complex K over $\mathbb{Z}$ is isomorphic to $\bigoplus_p \mathbb{Z} = \mathbb{Z}^p$, where p is the number of connected components of K.*

Proof. This follows from Lemma 1.14 and Definition 1.12. □

Here are a few examples:

Example. The 0th homology group of the circle is isomorphic to $\mathbb{Z}$.

To show this, think of the circle as being four 1-simplices. We have that Z_0 is the group consisting of sums over four 0-simplices, say, a, b, c, and d. Let x be some 0-chain $j_1 a + j_2 b + j_3 c + j_4 d$. Now let's reduce to an element of H_0. We do this by creating another chain $y = j_4 c - j_4 d$ and subtracting it from the previous one; we get

$$x - y = j_1 a + j_2 b + (j_3 - j_4)c.$$

Doing the same thing, we end up with a chain $z = (j_1 - j_2 + j_3 - j_4)a$. We have that $z \sim x$. We can now write $z = ja$, for $j \in \mathbb{Z}$. Therefore, for any j, we have $H_0 \cong \mathbb{Z}$.

Example. The homology group $H_n(S^n)$ is isomorphic to $\mathbb{Z}$.

We will not show this due to the length of the explanation, but the reader may see Nadathur's explanation on page 7 of her paper.

Example. The homology group $H_n(D^n)$ is equal to 0.

This is pretty easy to show. Let D^n be the simplicial structure of of the n-simplex (Δ^n). We have that every n-chain has the form $x = k\Delta^n$, for some $k \in \mathbb{Z}$. We have $H_n = Z_n$, because x is never a boundary. However, $\delta x = 0$ when $k = 0$. Therefore, $H_n(D^n) = 0$.

3. Singular Homology

Let's begin our study of singular homology with a few definitions. Although we cannot easily use the information we've already learnt and apply it to singular homology, the definitions here aren't too different.

Definition 3.1. A *singular n-simplex* in some topological space X is a map $\sigma : \Delta^n \to X$, where σ is continuous.

Definition 3.2. Let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices of X. We say that the elements of $C_n(X)$ are *singular n-chains* and are finite formal sums $\sum_i g_i\sigma_i$, where $g_i \in \mathbb{Z}$.

Definition 3.3. The *boundary map* $\delta_n : C_n(X) \to C_{n-1}(X)$ is given by

$$\delta_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\dots,v_{i-1},v_{i+1},\dots,v_n]},$$

where v_i are 0-simplices of σ.

We can now define the *singular homology group*:

Definition 3.4. The *singular homology group* $H_n(X)$ is the quotient $H_n(X) = \mathrm{Ker}(\delta_n)/\mathrm{Im}(\delta_{n+1})$.

Remark 3.5. To distinguish between the singular homology group and the simplicial homology group, we will write H_n^Δ for the simplicial homology group.

We won't go too in-depth with singular homology, but here are two propositions similar to those we've seen when dealing with general homology as in the previous section.

Proposition 3.6. *Let X be a topological space. Then*

$$H_n(X) \cong H_n(X_1) \oplus H_n(X_2) \oplus \cdots \oplus H_n(X_p),$$

where X_i are path-connected components of X.

Proof. A singular simplex has a path-connected image in X, because σ are continuous. This means we can write $C_n(X)$ as the following direct sum:

$$C_n(X) = C_n(X_1) \oplus \cdots \oplus C_n(X_p).$$

The boundary map δ is a homomorphism, so $\mathrm{Ker}(\delta_n)$ and $\mathrm{Im}(\delta_{n+1})$ split. Therefore, we have

$$H_n(X) \cong H_n(X_1) \oplus H_n(X_2) \oplus \cdots \oplus H_n(X_p),$$

which is what we wanted to show. □

Proposition 3.7. *The zero-dimensional homology group of a space X is the direct sum of copies of $\mathbb{Z}$, one for each path-component of X.*

Proof. See Nadathur's proof of her paper's Proposition 3.6. □

These propositions can be used to support the claim that $H_n^{\Delta} \cong H_n$. The formal statement is as follows:

Theorem 3.8. *For all n, the homomorphisms $H_n^{\Delta}(X) \to H_n(X)$ are isomorphisms*[7]*. Thus the singular and simplicial homology groups are equivalent.*

The proof of this theorem is not too difficult given more definitions, but we will not be covering these in this paper; see Nadathur's paper for more information and a proof.

4. Exact sequences

Let's now turn to exact sequences, an interesting topic in homology.

Definition 4.1. An *exact sequence* is a sequence of the form

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \ldots,$$

where the A_i are abelian groups, the α_i are homomorphisms, and $\mathrm{Ker}(\alpha_n) = \mathrm{Im}(\alpha_{n+1})$ for all n.

Definition 4.2. A *chain complex* is a sequence of abelian groups connected by homomorphisms (boundary operators) where the composition of any two consecutive maps is 0.

Now, let's relate exact sequences to chain complices.

Proposition 4.3. *An exact sequence is a chain complex, because*

$$Ker(\alpha_n) = Im(\alpha_{n+1}) \implies Im(\alpha_{n+1}) \subset Ker(\alpha_n) \iff \alpha_n\alpha_{n+1} = 0.$$

The homology groups of an exact sequence are trivial, because $Ker(\alpha_n) \subset Im_{\alpha_{n+1}}$.

Notice that there are different types of exact sequences: long and short.

[7]Here $H_n^{\Delta}(X) \to H_n(X)$, i.e., the homomorphisms from the simplicial homology group to the singular homology group

Definition 4.4. We say an exact sequence is *short* if it is of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

A *long* exact sequence

$$A_0 \xrightarrow{f} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \dots \xrightarrow{f_n} A_n$$

is an exact sequence consisting of more than three nonzero terms.

We have the following theorem, which involves exact sequences:

Theorem 4.5. *Let I and J be two ideals of some ring R. Then we have that*

$$0 \longrightarrow I \cap J \longrightarrow I \oplus J \longrightarrow I + J \longrightarrow 0$$

is an exact sequence of R-modules, where the module homomorphism $I \cap J \to I \oplus J$ maps every element $k \in I \cap J$ to the element (k, k) of $I \oplus J$, and the homomorphism $I \oplus J \to I + J$ maps each element $(k, l) \in I \oplus J$ to $k - l$.

Next, let's define a type of homology group called a *relative homology group*. We won't be going into this too much, but here is some background behind the definition. Given a space X and a subspace $A \subset X$, define $C_n(X, A)$ to be the quotient group $C_n(X)/C_n(A)$. The operator δ takes both $C_n(X) \to C_{n-1}(X)$ and $C_n(A) \to C_{n-1}(A)$.

Definition 4.6. We have that the following sequence

$$\cdots \longrightarrow C_{n+1}(X, A) \xrightarrow{\delta_{n+1}} C_n(X, A) \xrightarrow{\delta_n} C_{n-1}(X, A) \longrightarrow \dots$$

is a chain complex (because $\delta_{n+1}\delta_n = 0$). We say the homology groups $H_n(X, A)$ of this chain complex are *relative homology groups*. We have the following two properties of $H_n(X, A)$:

(1) Elements in the relative homology group are represented as *relative cycles*.[8]
(2) A relative cycle x is trivial if and only if it is a *relative boundary*.[9]

There is still another topic we will be covering, so we can't focus on relative homology, but I encourage the reader to read pages 11 to 13 of Nadathur's paper, mentioned earlier.

[8] *Relative cycles* are essentially n-chains x in $C_n(X)$ where $\delta_n(x) = C_{n-1}(A)$.

[9] In other words, if and only if x is the sum of a chain in $C_n(A)$ and the boundary of a chain in $C_{n+1}(X)$.

5. The Universal coefficient theorem

In this section, we will be looking at an interesting theorem in algebraic topology, called the *Universal coefficient theorem.* First, let's cover the basics.

Definition 5.1. There is a complex similar to the chain complex, namely, the *cochain complex*, which may be denoted $(A^*\delta^*)$. Essentially, the cochain complex consists of a sequence of abelian groups/modules $\dots, A^0, A^1, A^2, \dots$ connected by homomorphisms $\delta^n : A^n \to A^{n+1}$ such that $\delta^{n+1}\delta^n = 0$. We may write the following sequence to represent the cochain complex:

$$\dots \xrightarrow{\delta^{-1}} A^0 \xrightarrow{\delta^0} A^1 \xrightarrow{\delta^1} A^2 \xrightarrow{\delta^2} \dots.$$

Remark 5.2. Although the latter definition for the cochain complex is important, we won't be using it very technically (i.e., the notation is not extremely important).

We may wonder what is the main difference between chain complices and cochain complices. In chain complices, the dimension of the boundary operators decrease, and in cochain complices, the dimension of the boundary operators increase. After the next definition, we will be able to provide an alternative explanation.

Definition 5.3. Generally speaking, we say *cohomology* refers to a sequence of abelian groups in a topological space.

Getting back to the question, a difference between chain complices and cochain complices is that cohomology has a ring structure that homology lacks.

Definition 5.4. For a ring R, right R-module M, left R-module N, and abelian group G, we say that $\phi : M \times N \to G$ is an *R-balanced product* if the following conditions hold for $m, m' \in M$, $n, n' \in N$, and $r \in R$:

(1) $\phi(m, n + n') = \phi(m, n) + \phi(m, n')$.
(2) $\phi(m + m', n) = \phi(m, n) + \phi(m', n)$.
(3) $\phi(mr, n) = \phi(m, rn)$.

Definition 5.5. Let R be a ring. Let $R - \text{Mod}$ be the category of left R-modules, and let $\text{Mod} - R$ be the category of right R-modules. Fix a left R-module and denote it S. Consider some $T(A) = A \otimes_R S$ for $A \in \text{Mod} - R$. This has left functors L_iT. Now we define the Tor groups, for some i, by

$$\text{Tor}_i^R(A, S) = (L_iT)(A).$$

One more definition, and we'll be ready to look at the theorem.

Definition 5.6. For some ring R, a right R-module M, and left R-module N, we have that $M \otimes_R N$, the *tensor product*, is an abelian group with a universal balanced product

$$\otimes : M \times N \to M \otimes_R N.$$

By "universal," this means that for an arbitrary abelian group G and balanced product $g : M \times N \to G$, there is a homomorphism $g' : M \otimes_R N \to G$ where $g' \cdot \otimes = g$. Now we have the background information needed to state the theorem (for homology):

Theorem 5.1. *Consider the tensor product $H_i(X; \mathbf{Z}) \otimes A$. Then there is a short exact sequence*

$$0 \to H_i(X; \mathbf{Z}) \otimes A \xrightarrow{\mu} H_i(X; A) \to Tor_1(H_{i-1}(X; \mathbf{Z}), A) \to 0.$$

See Gallier and Quaintance's paper for a slightly different way of defining this theorem.

References

[GQ16] Jean Gallier and Jocelyn Quaintance. A gentle introduction to homology, cohomology, and sheaf cohomology. *Preprint*, 2016.

[Nad07] Prerna Nadathur. An introduction to homology. *University of Chicago*, 2007.