27 Lines on a Cubic Surface

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1 Introduction

We will define an algebraic surface as any surface which holds the form $f(x,y,z)=0$ where $f(x,y,z)$ is a polynomial in x,y,z. The order of the surface is equal to the degree of the polynomial. For this paper, we will focus on algebraic surfaces of order 3 which will be referred to as cubic surfaces. Our principle theorem will be the Cayley-Salmon theorem which states:

Theorem 1.1. *Given any smooth cubic surface over an algebraically closed field will contain exactly 27 straight lines.*

The proof for such theorem is several pages and cannot be condensed to fit into this paper without grossly straying for the main point. In a quick summary, Cayley that there were finitely many subspaces of all closed algebraic fields and Salmon proved that there were 27 2-dimensional subspaces of all closed algebraic fields. Depending on the field in question, this can be an easy task or quite the laborious one. We will look at a few specific examples later.

2 Examples of Cubic Surfaces

In his proof, Cayley gave the example of a nodal cubic surface defined by the equation:

$$wxy + xyz + yzw + zwx = 0 \tag{2.1}$$

This equation, however, only hold true in the complex field. This makes the notion that each line is straight and real a little more difficult to prove. The only real notable property of this surface is that its only automorphism group is S_4 of order 24. Because of it lack of exciting qualities, we will not be exploring this particular example any further.

While Cayley's nodal cubic space was the first surface that this new theorem was applied to, it was not the oldest example to fit the criteria. Fermat found the surface defined by:

$$x^3 + y^3 + z^3 = 1 \tag{2.2}$$

However, it the projective space, the surface is defined by:

$$w^3 + x^3 + y^3 + z^3 = 0 \tag{2.3}$$

The 27 lines on this cubic are relative easy to see and describe. The first subset of these lines take the form of $w : aw : y : by :$ for all fixed numbers a and b with cube -1. 9 of the 27 lines fall into this subset. The other 18 are conjugates of these lines under permutations of coordinates. While slightly more thrilling than the last example, there is not much more to be explored in terms of the 27 lines. So for this specific paper, we will be continuing on to another example.

The most notable example of this theorem comes from Alfred Clebsch who created his namesake space around the same time Cayley and Salmon released their theorem and proof. This surface is the most well researched surface in relation to the base theorem. The general cubic surface itself is defined by:

$$\mathbb{P}(\mathbb{R}^3) = \begin{cases} \mathrm{X}_2^3 + X_2^3 + X_3^3 + X_4^3 = 0 \\ \mathrm{X}_2 + X_2 + X_3 + X_4 = 0 \end{cases} \tag{2.4}$$

It is important to note that these constraints will only form the desired surface in the complex field. This surface has many fascinating features that we will take a closer look at.

3 The Clebsch Cubic

The Clebsch Cubic is studied greatly in part to the ease in which it can be written as function on a three dimensional plane. More specifically, the function

$$C = 81(x^3+y^3+z^3)-189(x^2y+x^2z+y^2x+y^2z_z^2x+z^2y)+54(xyz)+126(xy+xz+yz)-9(x^2+y^2+z^2)-9(x+y+z)+1 \tag{3.1}$$

can be obtained for the surface by considering the plane at infinity that is also within the upper bound of $x_0 + x_1 + x_2 + \frac{x_3}{2}$.

The Clebsch Surface is also commonly referred to as Clebsch's diagnoal surface. This is due to the fact that if any variable is dropped from the general cubic surface equation, the constraints are able to be factored out into the equation

$$(x+y)(x+z)(y+z) = 0 \tag{3.2}$$

Speaking of dropping variable from the general cubic surface equation, if x_0 were to be dropped, it becomes quite clear that the surface is isomorphic to the surface $(x_1 + x_2 + x_3)^3$ in P^3.

There is only once symmetry group of this surface, S_5 of order 120. A strikingly similar symmetry group to Cayley's surface which had a symmetry group of S_4 of order 24. These symmetry groups have no real meaning in terms of the base theorem, but the similarity seemed to non-coincidental for it not to be brought up in this paper.

3.1 The 27 Lines

The reason why the Clebsch surface is so well studied is due to the fact that the proof for the 27 lines is always non trivial. In order to best describe the lines, we will think of each line as a 2-dimensional subspace of $\mathbb{C}^5$. We will also define the function

$$\zeta = e^{\frac{2\pi i}{5}} \tag{3.3}$$

From these definitions, we are easily able to identify the 2-dimensional subspaces. 15 of the subspaces can be obtained by permuting the coordinates $(1, -1, 0, 0, 0)$ and $(0, 0, 1, -1, 0)$ in an arbitrary way. Similarly, the other 12 subspaces can be obtained by permuting the coordinates $(1, \zeta, \zeta^2, \zeta^3, \zeta^4)$ and $(1, \zeta^{-1}, \zeta^{-2}, \zeta^{-3}, \zeta^{-4})$. In order to prove that these subspaces lie in $S \subset \mathbb{C}^5$, we must consider the lemma specific to the Clebsch surface that states

Lemma 3.1. *All of the 27 lines are visible even if attention is restricted to only real solutions. This means only 2-dimensional real surfaces will be considered rather than complex surfaces.*

It then quickly follows that each subspace lies in $S \subset \mathbb{C}^5$.

4 Eckardt Points

Another area of interest on the Clebsch surface are the Eckardt points that lie on it.

Definition 4.1. An Eckardt point is the point of intersection between three concurrent lines of a surface.

A fun property of these points is that the only three lines on a surface are able to be classified as concurrent is if they lie in a tritangent plane.

Definition 4.2. A tritangent plane is a plane that intersects the surface in three lines.

The only classification for these planes is whether or not they are concurrent in any given point or not. By definition of a tritangent plane, there are only 45 which in turns mean s a surface will have at most 45 Eckardt points.

What makes the Clebsch surface unique in this regard is the fact that it is the only cubic surface that has exactly 10 Eckardt points. 7 of these points are able to be obtained visually after graphing the 27 lines. The other three lie at infinity. How can this be? Well we will use the lemma that

Lemma 4.3. *All lines inside tritanget planes fall into groups of three parallel lines.*

Given the fact that parallel lines intersect only at infinity and that the Clebsch surface yields three sets of parallel lines, we can conclude that there are precisely three more Eckardt points at infinity.

These points also have curious properties in being able to distinguish cubic surfaces up to isomorphism.

Theorem 4.4. *If two surfaces have a different number of Eckardt points, they are unable to be isomorphic. However, the converse is false in the fact that there are cubic surfaces with the same number of Eckardt points.*

5 The Double Six

Not long after the publication of the Cayley-Salmon theorem, a mathematician named Schläfli found and proved that the 27 lines are able to be described in terms of only 12 lines. The structure of these twelve lines are that:

Theorem 5.1. *Schläfli Double Six 6 of the lines are going to be pairwise skew or non intersecting are denoted as* $a_1, a_2, ..., a_6$*. The other 6 lines, also pairwise skew, will be denoted as* $b_1, b_2, ..., b_6$*. All of these lines must satisfy the property that* a_i *intersects* b_j *if and only if* $i \neq j$ *(i.e.* a_5 *intersects* b_3 *because* $5 \neq 3$*).*

This double six would be expressed as

$$\begin{array}{c} a_1, a_2, a_3, a_4, a_5, a_6 \\ b_1, b_2, b_3, b_4, b_5, b_6 \end{array} \tag{5.1}$$

Another important property of a double six is that they are what determine the surface of an equation. This is why they are commonly referred to as the "back-bone" of an equation surface. The 12 lines formed by the double six are able to give the general shape/outline of what the cubic surface will look like. This is especially helpful in the sense that it is much easier to draw 12 lines than to create a 3-D model of cubic surface. Double sixes allow for a sense of simplicity in a cubic surface.

It is important to note that there is not just a single double size among the 27 lines. In fact, given that two sets of 6 lines need to be chosen, there are 26 double sixes amount the 27 lines.

6 Other Configurations and Graphs

In order to look closer at what a double six of a Clebsch surface tells us, we must look at the bigger picture. A double six classifies twelve lines as being able to define the total 27 lines, but what happens to the excess 15 lines? Well, they are considered complimentary to the double six and when their properties are combined with the 15 tangent planes that run through these lines, it matches the Cremona-Richmond configuration.

Definition 6.1. The Cremona-Richard configuration is a configuration that consists of 15 lines and 15 points in which 3 points lie on each. line and three lines run through every point, yet no triangles exist.

It is quite easy to see that all of the Cremona-Richard configurations are subsets of a Schläfli double six configuration, given they are found using the definition above. Within the given 36 double sixes, it is guaranteed that there are 72 Cremona-Richard configurations. This is due to the fact that for every double six configuration on the Clebsch surface, there are exactly 2 Cremona-Richard configurations. There are not any more interesting properties in relation to the Clebsch surface, so we will leave extra research up to the reader.

Another interesting property of the Clebsch surface comes when looking at the intersection graph of the twelve lines of the double six configuration–commonly called the twelve vertex crown graph.

Definition 6.2. The twelve vertex crown graph is a bipartite graph where every vertex is adjacent to five out of the six vertices of the opposite set (i.e. a_i is adjacent to 5 points of b_j and vice versa).

Because this graph is able to be formed, this means that it is able to be proved that the Clebsch graph is isomorphic to the graph of cube. This exposes a plethora of possibilities for expository papers to be written about at a later date.

References

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