Algebraic curves

Lucas Perry

July 2020

Algebraic curves are the sets of points in a plane that are the zero of a polynomial in two variables.

1 Introduction

For example, the equation $y - x^2 = 2$ generates a parabola. $x^2 + y^2 - 1 = 0$ generates the unit circle. Another well known example is the hyperbola $xy - 1 = 0$. These are all conics, and so is the ellipse. These are all equations of degree 2, i.e. involving terms x^2,y^2,and xy. It may also use terms of lower degree, i.e. x, y, and a constant term such as +4. Therefore the general form for any conic in the xy plane is $ax^2 + by^2 + cxy + dx + ey + f = 0$. (For it to really be a conic requires that at least one of a, b or c is not zero.) For example, the equation for an ellipse $(x - h)^2/a^2 + (y - k)^2/b^2 = 1$ derives itself from this. But what happens when there are higher degree terms, such as x^3 or x^2y^3? How does this change the behavior of the curve, and what can be used to detect these patterns? The answer to this question lies in a study of the intersections of the curve on itself, namely singularities.

2 Cubics

Just as there is a quadratic equation $y = ax^2 + bx + c$, there is a cubic equation of similar form $y = ax^3 + bx^2 + cx + d$. It is apparent how this translates to a zero of polynomial in two variables: $ax^3 + bx^2 + cx + d - y = 0$. We are not thinking of all x such that y=0 where this is satisfied, but instead all ordered pairs (x,y) where this is so. If we were to translate this to an all-encompassing form for polynomials in two variables of degree 3, it would look something like this: $a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 + a_5x^2 + a_6xy + a_7y^2 + a_8x + a_9y + a_{10} = 0$ where $a_1, ..., a_{10}$ are all real constants. Notice that this encompasses all conics, that is when $a_1, a_2, a_3, a_4 = 0$, but we will only deal with situations where at least one of these in nonzero.

3 How to find singularities

Given the curve $x^3 - y^3 + xy = 0$. It is clear that (0,0) satisfies here. As a point on the curve approaches (0,0), both higher degree terms will be much smaller than xy. So the general shape around the origin is xy=0, which is equivalent to the x and y axis. At this point the graph crosses over itself, so this is a node.

It is important to note that, in the case that the curve is unbounded in the real numbers, the opposite effect will be achieved when the variables are made very large. When x and y here are made to be very large, the term xy will be much smaller in value than x^3 and y^3, and therefore can be ignored. The asymptote of the curve becomes $x^3 - y^3 = 0$ or $x = y$. The larger the value of x gets, the closer it will be in value to y.

4 Algebraic curves in higher dimensions

It is possible to conceive of curves in a 3-dimensional space. For example, $x^2+y^2+z^2 = r^2$, where r is the radius of the sphere. Or $x^2/a^2+y^2/b^2+z^2/c^2 = 1$ is a spheroid. Or a plane such as $x + 2y - 4z - 5 = 0$. Because a line is a one dimensional object in a three dimensional space, it is representing as 3 expressions, such as $x = y = z$. Another nice 3-dimensional curve is the cone, which has an equation $x^2 + y^2 - z^2 = 0$. It is easy to see that at every point along the z axis, there is a circle around that point, perpendicular to the z axis, of radius $|z|$, described by $x^2 + y^2 = z^2$, the standard equation for a circle in 2 dimensions. But we will not dwell too long on algebraic curves of higher dimensions because they are harder to conceive of, hard to reason with, and hardly fit the definition of algebraic curve, which is normally restricted to two dimensions.

5 Constructing certain curves

A common coincidence in two-variable polynomials is superimposing. To "superimpose" two graphs of equations set equal to zero on to each other, just multiply them. For example, as seen earlier, xy=0 is the set of all point that lie either on the x-axis OR on the y-axis. This is because only one of the two factors x,y has to be equal to 0 for the whole expression to be equal to 0. Similarly, to superimpose the line y=1 on the parabola $y = x^2$, describe $(y - 1)(y - x^2) = 0$. This can work for any number of curves at a time.

Sometimes it is the case that a polynomial in two variables really can be factored in this way.

Another method of factoring is using the common quadratic formula, in the case that one of the variables never exceeds a power of 2. For example, $x^2 - 2y^2 + xy = 0$ can be factored as $(x + 2y)(x - y)$, which is just two lines superimposed on each other.

Or, to factor the equation $x^2 - xy^2 + y^3 = 0$ into $(x - a)(x - b)$, we treat y

not as a variable but instead as a constant, and obtain two roots: $\frac{-y^2+\sqrt{y^4-4y^3}}{2}$ and $\frac{-y^2-\sqrt{y^4-4y^3}}{2}$. This essentially gives us two halves of the graph, a result of the two equations superimposed on each other: $x = \frac{-y^2+\sqrt{y^4-4y^3}}{2}$ and $x = \frac{-y^2-\sqrt{y^4-4y^3}}{2}$. This is similar to dividing the graph of $y^2 = x$ into $y = +\sqrt{x}$ and $y = -\sqrt{x}$. This process can sometimes tell us about the number of real solutions, such as in solving for x in $x^2+y^2+xy = 0$, which tells us that the only real solution is (x,y)=(0,0). It also makes it easier to differentiate or integrate, but it is possible to use implicit differentiation and obtain the same result. This can help us detect special points, write the equation of a tangent line, etc...

6 Differentiating algebraic curves

Implicit differentiation of algebraic curves is very easy. Differentiate in the same way as in one variable, using the power rule and multiplication rule, except multiply by dx and dy for x and y terms respectively.

For example, to differentiate $x^2+y^2-3xy+4 = 0$, we obtain $2xdx+2ydy-3(xdy+ydx) = 0$. Then we can isolate dy/dx in terms of x and y to obtain the slope of the tangent line at a given point (x,y). Here it is $dy/dx = x/y$. At the point (4,2) the slope is 4/2=2.

7 Properties of algebraic curves

There are various properties that are easy to look for. One is symmetry. If a polynomial in two variables contains all even degree terms or all odd degree terms, then it is symmetric about the origin (this doesn't necessarily mean it contains the origin). Note: constants count as even degree because there are ax^0y^0.

This tells us why $y - x^3 = 0$, $xy - 1 = 0$, and $x^2 + y^2 - 1 = 0$ are symmetric about the origin. Similarly, if a polynomial in two variables contains all even y terms, then it will be symmetric about the x-axis, and vice-versa. In other words if a polynomial can be expressed as a polynomial in x and y^2, then it is symmetric about the x-axis. A polynomial in x^2 and y^2 is symmetric over BOTH axes. This tells us why $y^2 - x^3 = 0$ behaves this way too. This can be expressed as $y = x^{3/2}$. Because $x^{3/2} < x$ for every $x < 1$, $x^{3/2}$ will skim the x-axis near 0, and so will the reflected curve. Therefore, these meet at a sharp point at (0,0).

8 Conclusion

In Conclusion, some algebraic curves can be described using very simple algebra. These are only a few examples of ways to describe algebraic curves using simple algebra, but there is much to be learned from these techniques in describing the behavior of Algebraic curves.