# GERBALDI'S THEOREM ON QUADRATIC FORMS

A FASCINATING AND EXQUISITE THEOREM ON TERNARY QUADRATIC FORMS IN PROJECTIVE GEOMETRY


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# Contents

# §1 Preface

In this expository paper, the end goal is provide a proof Gerbalid's theorem. He claimed the existence of six mutually apolar linearly independent nondegenerate ternary quadratic forms. Throughout this paper, we familiarize the key concepts and definitions required to fully comprehend the proof. To commence, we offer some basic abstract algebraic definitions on groups and fields and then we move onto some linear algebra. Finally, we introduce the notion of a bi-algebra, a crucial structure in the field of mathematics. Of course, since Gerbalid's theorem is a part of the realm of projective geometry, it is essential to quickly introduce the intricate world of projective geometry. For a bit of more interingness, we proceed to state 2 integral theorems on conics, Pascal's Hexagon theorem and one of its special cases, Pappus' theorem. Both are easily proven via Bezout's theorem and we let the reader attempt to prove these two on their own. Finally, to conclude the paper, we present the desired 6 quadratic forms in its monomial and reciprocal form as well.

I really enjoyed writing this paper because it really opened my eyes to the real world of mathematics, beyond the confinements of high school math where only BC calculus is the "roof". Of course, I had the oppurtunity to learn about various topics of mathematics, such as projective geometry, abstract algebra, linear algebra and even algebraic geometry. I am very grateful for this learning oppurtunity .

There are a few people that have helped me out throughout this journey of writing this paper and I am very appreciative of their help. Of course, to begin, I must thank Simon Rubinstein-Salzedo for being a great instructor and helping me so much during the problem sessions. He has helped me numerous times with LaTeXerrors as well. Without him, this paper would be pretty darn bad. Next, I have to thank Tyler Shibata, for being such an awesome TA. He has helped me throughout the course with my problem sets and in this paper, has also provided valuable feedback that I have implemented after the draft.

Without further ado, let us begin with our paper.

# §2 Preliminary Definitions

These definitions form the fundamentals of abstract algebra but are quite basic and elementary as well. Experienced readers are not required to read through this section and are encouraged to skip to the part where we start defining a "bi-algebra". However, if the reader is not acquainted with linear algebra, it is recommended that you go over some of the preliminary definitions from the linear algebra subsection.

## 2.1 Group Theory and Fields

**Definition 1** *Groups:*

*A group(non-abelian) is a set G with both a binary operation $\cdot$: $G \times G \to G$(which means that for any $g, h \in G$, we have a well-defined element $g \cdot h$ $\in$G), with the following properties:*

- *There is an element $e \in G$ so that $eg = ge = g$ for all $g \in G$(We call e the identity element of G)*
- *If $g, h, k \in G$, then $g(hk) = (gh)k$ (associativity property).*
- *For every $g \in G$, there is some element $h \in G$ so that $gh = hg = e$. We call$h$ an inverse of $g$.*

An abelian group is just a group with one further property, described below.

**Definition 2** *Abelian Groups:*

*A group is said to be "abelian" if for all $x, y \in G, x \cdot y = y \cdot x$*

**Example 1** *The most simple example of an abelian group is a cyclic group, which are groups generated by a single element, where the cyclic group of order $n$ is the group of integers modulo $n$.*

Now, it is time to define a **field**. Essentially, it is a mathematical structure with a few properties which are encompassed below:

**Definition 3** *Fields:*

*A field is a set F, together with two binary operations $+$ and $\times$ (which are called addition and multiplication), so that all the following properties hold:*

1. *F forms an abelian group under addition. We call the identity element with respect to addition 0.*
2. *$F^{\times}$ = F $\{0\}$ forms an abelian group under multiplication. We call the identity element with respect to multiplication 1.*
3. *The distributive property holds: $a(b + c) = ab + ac$.*

**Example 2** *The rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$ are all fields but the integers $\mathbb{Z}$ do not form a field since all non-zero elements do not form a group under multiplication since they do not have multiplicative inverses.*

## 2.2 A Few Linear Algebra Definitions

Now, it is time for a quick crash course in **linear algebra**. These are the only definitions we need from linear algebra in order to comprehend the proof of our main theorem. Note that linear algebra itself is another elegant subject in its own entirety and to properly learn the basics of linear algebra, it would take months of studying!
Now, we present another extremely mathematical structure, which forms the fundamentals of linear algebra.

**Definition 4** *Vector Space:*

*Let F be a field. A vector space over F is an abelian group V, formed with a binary operation $\cdot : F \times V \to V$, which is called scalar multiplication, so that for all $a, b \in F$ and $v, w \in V$, the following holds:*

- $0_F v = 0V$ *, where $0_F$ is $0$ in $F$ and $0_V$ is the identity of $V$ ,*
- $1_v = v$*,*
- $a(v + w) = av + aw$*,*
- $(ab)v = a(bv)$*,*
- $(a + b)v = av + bv$

Next, we need to know what a **k-vector space** is. It is essentially the same as a the vector space defined above but with a few extra properties.

**Definition 5** *k-Vector Space:*

*Let **k** be an arbitrary field. A k-vector space is an abelian group $(V, +)$, equipped with an external operation*

$$\boldsymbol{k} \times V \ni (\lambda, v) \to \lambda v \in V$$

*(once again, called scalar multiplication), with the following properties:*

- $\lambda \cdot (v + w) = (\lambda \cdot v) + (\lambda \cdot w)$*, for all $\lambda \in$ **k** and $v, w \in V$(distributive property)*
- $(\lambda + \mu)\Delta v = (\lambda v) + (\mu v)$*, for all $\lambda, \mu \in k, v \in V$ (distributive property).*
- $(\lambda \cdot \mu)v = \lambda \cdot (\mu \cdot v)$*, for all $\lambda, \mu \in k$ and $v \in V$ (associative property)*
- $1 \cdot v = v, forallv \in V$*( identity element is 1)*

Now, we define **linear independance, spans and basis** . Indeed, these are definitions that can be found in any linear algebra book and aer fairly intuitive and straightforward.

**Definition 6** *"Linear Independance, Spans and Basis":*

*Let F be a field and V a vector space over F.*

- *We say that* $v_1, ..., v_n \in V$ *are linearly independent if, whenever we have* $a_1, ..., a_n \in F$ *with* $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$*, then* $a_1 = a_2 = ... = a_n = 0$.
- *We say that* $v_1, ..., v_n \in V$ *span V if, for every* $v \in V$ *, there are elements* $a_1, ..., a_n \in F$ *such that* $v = a_1v_1 + a_2v_2 + ... + a_nv_n$.
- *We say that* $v_1, ..., v_n \in V$ *form a basis for V if they are linearly independent and span* $V$.

**Example 3** $\{1, x, x^2\}$ *is a basis of* $\{ax^2 + bx + c|a, b, c \in \mathbb{R}\}$

### 2.3 Bialgebras

Now, we are finally ready to define a bi-algebra. Here is when all the fun starts but hold on tight-since these definitions are also quite hard to grasp at first as well.

**Definition 7** *Bialgebra:*

*Let W be a finite dimensional k-vector space. Then a bialgebra consists of a bimultiplication law on W(a structure on W) which consists of 2 bilinear maps(which we will denote by square brackets[]),*

$$W \times W \to W^*, (a, b) \mapsto [a, b] \in W^*$$

$$W^* \times W^* \to W, (p, q) \mapsto [p, q] \in W$$

## §3 Conventions

Throughout this expository paper, there are several conventions that will be used. Firstly, the ground field K is always characterisic zero and algebraically closed. We offer some definitions belows to ensure that the audience understands:

**Remark 1** ***Ternary Quadratic Forms*** *are polynomials in three variables with all terms of degree 2.*

**Example 4** $x^2 + xy + yz + xz + z^2$ *is a ternary quadratic form.*

**Definition 8** *Characteristic:*

*Let R be a ring. The characteristic of a ring R, denoted by char(R), is defined to be the smallest number of times one must use the ring's multiplicative identity in a sum to get the additive identity.*

The ring $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n has characteristic $n$. (Check this!)

**Definition 9** *Algebraically Closed:*

*A field $K$ is said to be algebraically closed if every polynomial with coefficients in $K$ has a root in $K$.*

Let $W$ be a finite-dimensional K-vector space, we denote $W^*$ to be the dual space of W. Further, if define a two-dimensional subspace of $W$ or $W^*$ to be a *pencil*. If $\{a, b\}$ is a basis of a pencil P, then we will denote the pencil $P$ by $(\lambda + \mu b)$ Finally, let

$$W \times W^* \to K, (a, p) \mapsto < a, p > \in K$$
$$W^* \times W \to K, (p, a) \mapsto < p, a > \in K$$

be two natural pairings denoted with the same symbol $<, >$.
Further, we say that two bases $e_1, ..., e_n \subset W, f_1, ..., f_n \subset W^*$ are *projectively dual*, if

$$< e_1, f_j >= 0 \text{ for } \neq j, \text{ and } < e_i, f_1 > \neq 0$$

# §4 A Quick Introduction to Projective Geometry

Projective geometry is another intricate world by itself but to keep it simple and easy to understand, projective geometry is just an extension of Euclidean Geometry but with a few extra characteristics. First, we add *points at infinity* for every set of parallel line. Thus, for every pair of parallel lines, they would no longer never intersect. Instead, those lines would now intersect at the *point at infinity*. Not only, we add a new projective line to the real affine space, the *line at infinity* which passes through all the points at infinity.

## 4.1 Axioms of Projective Geometry

These axioms form the basis (no pun intended) of projective geometry. These axioms may sound quite simple at first but still play an integral role in the world of projective geometry.

**Axiom 1** *Whitehead's Axioms*

1. *Every line contains at least 3 points*
2. *Every two distinct points, A and B, lie on a unique line, AB.*
3. *If lines AB and CD intersect, then so do lines AC and BD (where it is assumed that A and D are distinct from B and C).*

### 4.2 The Concept of Projective Duality

The most elegant part of projective geometry is none other than the concept of projective duality. In short, given any theorem or definition in projective geometry, we can substitute the words *point* for *line*, *lie on* for *pass through*, *collinear* for *concurrent*, *intersection* for *join*, or vice versa, which results in another theorem or valid definition, the "dual" of the first.
More formally, we have the following:

**Proposition 1** *A projective plane C may be defined axiomatically as an incidence structure, in terms of a set $P$ of points and a set $L$ of lines, and an incidence relation I that determines which points lie on which lines.Now, for the concept of duality:*

*We can interchange the role of points and lines in $C = (P, L, I)$ to obtain its dual structure: $C^* = (L, P, I^*)$, where I* is the converse relation of I.*
*We call $C^*$the dual plane. It also follows by definition that $C^{**} = C$, which can be thought of as an involution.*

**Example 5** *(The dual of Whitehead's Axiom #2)*

*Every 2 distinct lines, $l_1$ and $l_2$, pass through(intersect) at a unique point, A.*
*The proof is trivial and follows immediately by adding a point at infinity. Indeed, if the lines are parallel, they intersect at the point at infinity. If the lines are not parallel, then the result follows from "regular" Euclidean Geometry.*

### 4.3 More Definitions in Projective Geometry

**Definition 10** *Apolarity*

*Two elements $a, b \in W$ of an arbitrary symmetric bialgebra (W,[,]) are mutually apolar, if and only if $< a, \hat{b} >=< b, \hat{a} >= 0$. Futhermore, $a, b$ are called nondegenerate mutually apolar elements, if they are mutually apolar and*

$$a \neq 0, b \neq 0, \hat{a} \neq 0, \hat{b} \neq 0, \hat{\hat{a}} \neq 0, \hat{\hat{b}} \neq 0$$

**Definition 11** *Projective Space*

*Let $k$ be a field. The $n$-dimensional* ***projective space*** *over $\mathbb{P}^n_k$ over $k$ is defined to be the set of lines passing through the origin in $\mathbb{A}^{n+1}_k$*

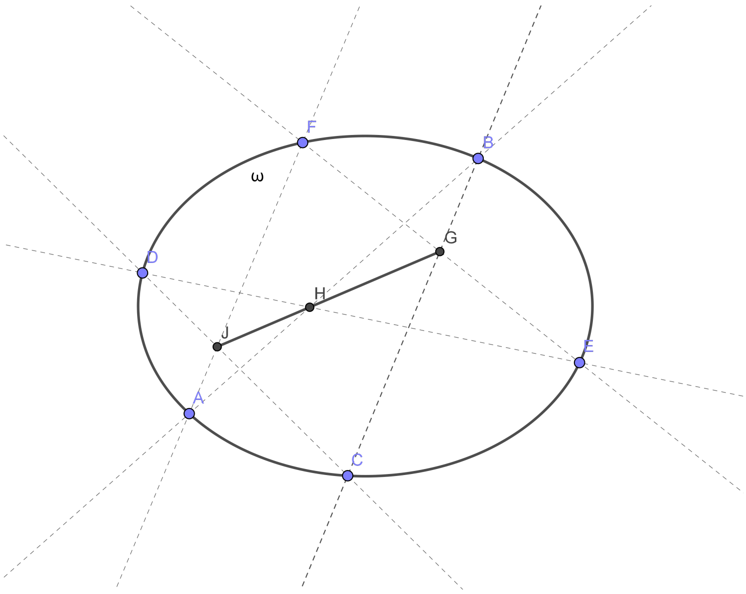


Figure 1: 6 points on a conic and Pascal's Theorem

# §5 An Interesting Digression

One of the most special parts of projective geometry is arguably the geometry theorems that are not onyl applicable in projective geometry but also in Euclidean geometry. For example, the famed **Pascal's Theorem** is an elegant theorem on 6 points on a circle, which form a hexagon.

## 5.1 Two Theorems on Conics

**Theorem 12** *Pascal's: Let* $A, B, C, D, E, F$ *be 6 points on a circle(conic section). Then* $H = AB \cap DE, G = BC \cap EF$ *and* $J = CD \cap FA$ *are collinear.See figure 1.*

**Theorem 13** *Pappus': If* $A, B, C$ *are three points on a line* $\ell_1$*, and* $A', B', C'$ *are three points on a line* $\ell_2$*. If the lines* $AB'$ *and* $A'B$ *meet at* $R$*, the lines* $BC'$ *and* $B'C$ *meet at* $P$*, and the lines* $CA'$ *and* $C'A$ *meet at* $Q$*, then the points* $P, Q$ *and* $R$ *are collinear. See figure 2.*

The morally correct way to prove these two theorems is via Bézout's Theorem. Indeed, there are many different proofs of the Pascal's hexagon Theorem such as using Menelaus repeatedly. However, this does not accurately represent the derivation of this beautiful theorem since these types of proofs require many clever insights and may seem very unnatural. For the sake of completeness, we state the prominent **Bezout's Theorem**.

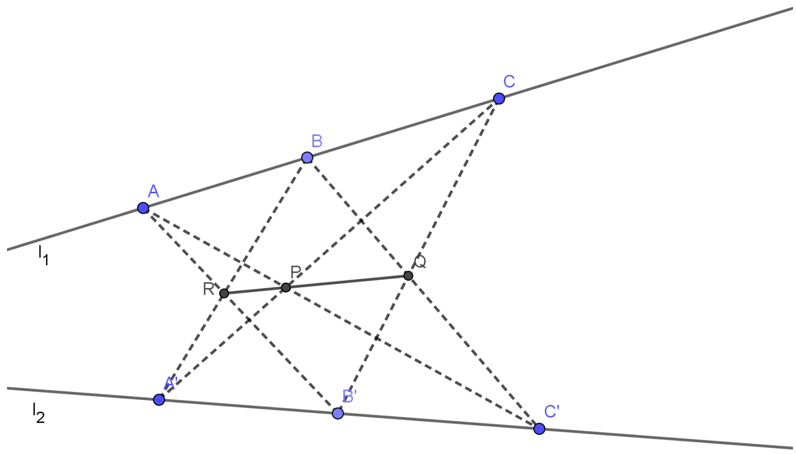


Figure 2: 2 lines and 6 points, an intriguing collinearity

### 5.2 Understanding Bézout

**Definition 14** *Intersection Mulitplicity:*

*Let $C_1 = V(f)$ and $C_2 = V(g)$ be 2 curves in $\mathbb{P}^2_{\mathbb{C}}$ that intersect at a point $p \in \mathbb{P}^2_{\mathbb{C}}$. Let $R_p$ be the subring of $\mathbb{C}[x, y, z]$ such that $s$ and $t$ are homogeneous of the same degree and such that $t(p) \neq 0$. Then the* ***intersection multiplicity*** *$m$, $i(p, f \cap g)$, is defined to be the dimension of the ring $R_p/(f, g)$ as a complex vector space*

**Theorem 15** *Bezout's:*

*Let $f$ and $g$ be homogeneous polynomials in $\mathbb{C}[x, y, z]$ of degrees $d$ and $e$, respectively, such that $V(f)$ and $V(g)$ share no common component(factor). Then*

$$\sum_{p \in \mathbb{P}^2_{\mathbb{C}}} i(p, f \cap g) = d \cdot e$$

*where $i(p, f \cap g)$ is defined to be intersection multiplicity, which is the dimension of the ring $R_p/(f, g)$ as a complex vector space.*

### 5.3 A Proof Using Resultants

Indeed, there exists several methods of proving Bezout's theorem but some require much more knowledge than others. For example, there exists a method involving Hilbert polynomials. We offer a more natural and hands-on proof but it may also seem quite long so we omit lots of calculations as well and treat it more like a "walkthrough".
To commence, we require the notion and knowledge of the **resultant**

**Definition 16** *The resultant Res (f,g) of 2 polynomials* $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ *and g(x) =* $g(x) = b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0$ *is defined to be the determinant of the (m+ n) ( m+ n) matrix*

$$Res(f,g) = \det \begin{pmatrix} a_n & a_{n-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_n & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n & a_{n-1} & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_m & \cdots & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_m & b_{m-1} & \cdots & b_0 \end{pmatrix}$$

**Proposition 2** *The resultant is zero precisely when the 2 polynomials have a common root.*

Now, using 3 lemmas, we can deduce Bezout's theorem. We prove none of them and leave them due to length.

**Lemma 1** *Let* $f, g \in \mathbb{C}[x, y, z]$ *be homogeneous polynomials of degrees* $m$ *and* $n$*, respectively. If* $f(0, 0, 1)$ *and* $g(0, 0, 1)$ *are nonzero, then Res(f,g;z) is homogeneous degree* $mn$ *in* $x$ *and* $y$

**Lemma 2** *Let* $f \in \mathbb{C}[x, y]$ *be homogeneous, and let* $V(f) = \{(r_1, s_1), ..., (r_t, s_t)\}$*. Then*

$$f = c(s_1 x - r_1 y)^{m_1} ...(s_t x - r_t y)^{m_t}$$

*where c is a nonzero constant*

**Lemma 3** *Let* $V(f)$ *and* $V(G)$ *be curves in* $\mathbb{P}^2_{\mathbb{C}}$ *so that the point (0,0,1) is not in* $V(f)$ *or* $V(g)$ *and is not collinear with any two points of* $V(f) \cap V(g)$*. Then if* $p = (u : v : w)$ *is in* $V(f) \cap V(g)$*, then* $I(p, V(f) \cap V(g))$ *is the exponent of* $(vx - uy)$ *in the factorization of* $Res(f, g; z)$

Indeed, it would be quite difficult to deduce Bézout's theorem using these 3 lemmas. If in need of help, check out [**6**] for a formal proof which includes computations as well.

# §6 Main Theorem

Now, we are ready to state the most significant and pertinent theorem of this expository paper.

## 6.1 Gerbadi's Theorem

**Theorem of Gerbaldi:**
There exists a set of six mutually apolar linearly independent nondegenerate ternary quadratic forms.

The main proof consists of Gerbaldi's list of six conics. This proves the existence of the theorem.

First, we fix a few constants which will be used in Gerbaldi's list. Let

$$\omega = exp(\frac{2\pi i}{3}) = \frac{-1+i\sqrt{3}}{2}, r = \frac{-\sqrt{3}+i\sqrt{5}}{4}, s = \frac{1+i\sqrt{15}}{4}$$

The comples conjugates or these numbers are

$$\bar{\omega} == \frac{-1-i\sqrt{3}}{2}, \bar{r} = \frac{-\sqrt{3}-i\sqrt{5}}{4}, \bar{s} = \frac{1-i\sqrt{15}}{4}$$

## 6.2 Gerbaldi's List

Consider the following six ternary quadratic forms $f_1, ..., f_6$

$$f_1 = x_0^2 + 2cx_1x_2$$
$$f_2 = \omega(x_1^2 + 2cx_0x_2)$$
$$f_3 = x_2^2 + 2cx_0x_1$$
$$f_4 = -\frac{1}{3}(1+2c)\omega[x_0^2 + x_1^2 + x_2^2 - c(x_1x_2 + x_2x_0 + x_0x_1)]$$
$$f_5 = -\frac{1}{3}(1+2c)[x_0^2 + \omega^2 x_1^2 + \omega x_2^2 - c(x_1x_2 + \omega^2 x_2x_0 + \omega x_0x_1)]$$
$$f_6 = -\frac{1}{3}(1+2c)\omega[x_0^2 + \omega x_1^2 + \omega^2 x_2^2 - c(x_1x_2 + \omega x_2x_0 + \omega^2 x_0x_1)]$$

Checking for mutual apolarity and linear independence is merely just a straight-forward computation. However, to check for mutual apolarity, we also need the reciprocal forms of the 6 ternary quadratics forms listed above. The expressions of the basic quadratic monomials in terms of the 6 forms are also listed below as well.

**The expressions of the basic quadratic monomials in terms of the 6 ternary quadratic forms $f_1, f_2, ..., f_6$**

$$x_0^2 = \frac{1}{9}(3f_1 - 2(1-c))(\omega^2 f_4 + f_5 + \omega^2 f_6)$$
$$x_1^2 = \frac{\omega^2}{9}(3f_2 - 2(1-c))(f_4 + \omega^2 f_5 + \omega^2 f_6)$$
$$x_2^2 = \frac{1}{9}(3f_3 - 2(1-c))(\omega^2 f_4 + \omega^2 f_5 + f_6)$$
$$x_1x_2 = \frac{1}{9c}(3f_1 - 2(1-c))\omega^2 f_4 + f_5 + \omega^2 f_6)$$
$$x_2x_0 = \frac{\omega^2}{9}(3f_1 - 2(1-c)(f_4 + \omega^2 f_5 + \omega^2 f_6)$$
$$x_0x_1 = \frac{1}{9c}(3f_1 - 2(1-c))(\omega^2 f_4 + \omega^2 f_5 + f_6)$$

**Reciprocals of the six ternary quadratic forms $f_1, ..., f_6$:**

$$\hat{f}_1 = -c(cu_0^2 + 2u_1u_2)$$

$$\hat{f}_2 = -c\omega^2(cu_1^2 + 2u_0u_2)$$

$$\hat{f}_3 = -c(cu_2^2 + 2u_0u_1)$$

$$\hat{f}_4 = \frac{1}{3}(c+2)\omega^2[c(u_0^2 + u_1^2 + u_2^2) - (u_1u_2 + u_2u_0 + u_0u_1)]$$

$$\hat{f}_5 = \frac{1}{3}(c+2)[c(u_0^2 + \omega u_1^2 + u_2^2) - (u_1u_2 + u_2u_0 + \omega^2 u_0u_1)]$$

$$\hat{f}_6 = \frac{1}{3}(c+2)\omega^2[c(u_0^2 + \omega^2 u_1^2 + \omega u_2^2) - (u_1u_2 + \omega^2 u_2u_0 + \omega u_0u_1)]$$

Thus, we have proven the existence of the 6 ternary mutually apolar linearly independent non-degenerate ternary quadratic forms, which was exactly what we wanted.

# §7 Parting Words

Projective Geometry is one of the major mathematical accomplishments of the nineteenth century and remains an intricate jewel to this date. Indeed, it has uses from not only other areas of mathematics but is also applicable to real life situations such as engineering, computer vision and computer graphics.

Firstly, projective geometry forms the basis of algebraic geometry and is basically a *prerequisite*. As we have seen in this paper, there are many elegant theorems which are proven by using projective geometry as an implicit step, also serving as a space where Bezout's theorem holds. Evidently, Bezout's theorem would not hold in regular affine space.

Further, **Gerbaldi's Theorem** also plays an intriguing role/application in group theory as well, the existence of the Valentiner group. In short, the *Valentiner group* is the prefect triple cover of the alternating group(the group of even permutations of a finite set) on 6 points which was proven in the form of an action $A_6$ on the complex projective plane. Gerbaldi showed that this group(the alternating group $A_6$) also acts on the 6 conics of Gerbaldi's Theorem, which was listed above. However, this is well beyond the scope of this expository paper and much more sophisticated knowledge of mathematics is required.

Projective geometry also plays an integral role in camera calibration in which projective transformations(an automorphism of $\mathbb{P}_k^n$) give a simple explanation of the perspective(pinhole) model of a camera.

Another application is in computer graphics, where it is necessary to convert the 3D coordinate world of point into a two-dimensional view plane, which is also achieved by using a projective transformation.

While these have been extremely brief descriptions of applications of projective geometry in the real world, the author urges the interested readers to learn more about this on other more thorough sources.

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