RIEMANN-HURWITZ THEOREM

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1 Introduction

In this paper, we mainly introduce the Riemann-Hurwitz formula. Named after Bernhard Riemann and Adolf Hurwitz, Riemann-Hurwitz formula describes the relationship of the Euler characteristics of two surfaces when one is a ramified covering of the other. In section 2, we mainly introduce some preliminary knowledge that are necessary for the understanding of Riemann-Hurwitz formula. Subsection 2.1, 2.2, and 2.3 presents some basic concepts in complex analysis, such as complex differentiation, complex charts, complex structures, manifolds, and Riemann surfaces. Subsection 2.4 and 2.5 focus on algebraic topology side, introducing some interesting operations on surfaces such as orientation, triangulation, and coverings. Finally, in section 3, we present the main statement of Riemann-Hurwitz formula.

2 Preliminaries

Definition 2.1. A *homeomorphism* is a continuous bijective function between topological spaces with a continuous inverse function.

Example 2.1. The open interval (a, b) is homeomorphic to $\mathbb{R}$ for any $a < b$.

Nonexample 2.1. $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic when $m \neq n$.

2.1 Complex Differentiation

As we are probably familiar with the differentiation of real valued functions, we now introduce the notion of complex differentiability.

Definition 2.2. A function $f : \Omega \to \mathbb{C}$, with $\Omega \subset \mathbb{C}$ open, is called *complex differentiable at* $z_0 \in \Omega$, if there is a function $g : \Omega \to \mathbb{C}$, which is continuous at z_0, such that

$$f(z) = f(z_0) + g(z)(z - z_0)$$

for all $z \in \Omega$. We call the value $g(z_0)$ the *derivative of f at* z_0, and write

$$f'(z_0) = \frac{df}{dz}(z_0) = g(z_0).$$

The following lemma gives a sequential criterion of complex differentiability.

Lemma 2.3. Let $\Omega \subset \mathbb{C}$ be an open subset. A function $f : \Omega \to \mathbb{C}$ is complex differentiable at z_0 if and only if there exists a number $\lambda \in \mathbb{C}$ such that

$$\lim_{n\to\infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} = \lambda \qquad (2.1)$$

for every sequence $\{z_n\} \subset \Omega$ converging to z_0.

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Proof. Suppose that f is complex differentiable at z_0. Then by definition 2.2, we have

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

for $z \in \Omega \setminus \{z_0\}$. Since $g(z)$ is continuous at z_0, for any sequence $\{z_0\} \subset \Omega$ converging to z_0, we have

$$\lim_{n\to\infty} g(z_n) = \lim_{n\to\infty} \frac{f(z) - f(z_0)}{z - z_0} = g(z_0) = \lambda.$$

For another direction, suppose that there exists $\lambda \in \mathbb{C}$ such that (2.1) holds for every sequence $\{z_0\} \subset \Omega$ converging to z_0. Then we define a function $g : \Omega \to \mathbb{C}$ by

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

for $z \in \Omega \setminus \{z_0\}$, and $g(z_0) = \lambda$. This function is continuous at z_0, and by construction satisfies $f(z) = f(z_0) + g(z)(z - z_0)$. ■

Let us look at some explicit examples of complex differentiation.

Example 2.2. Consider $f(z) = z^2$, and its differentiability at some point $z_0 \in \mathbb{C}$. We have

$$f(z) - f(z_0) = z^2 - z_0^2 = (z + z_0)(z - z_0),$$

and hence $f(z) = f(z_0) + g(z)(z - z_0)$ with $g(z) = z + z_0$. Then function $g(z) = z + z_0$ is clearly continuous at z_0, meaning that f is complex differentiable at z_0, with

$$f'(z_0) = g(z_0) = 2z_0.$$

Nonexample 2.2. Now let us consider $f(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z. Let $z_0 \in \mathbb{C}$, and take $z_n = z_0 + h_n$, where $\{h_n\} \subset \mathbb{R}$ is a real sequence converging to 0. Then we have

$$f(z_n) - f(z_0) = \overline{z_n} - \overline{z_0} = \overline{z_0} + \overline{h_n} - \overline{z_0} = \overline{h_n},$$

and $z_n - z_0 = h_n$, which implies that

$$\frac{f(z_n) - f(z_0)}{z_n - z_0} = \frac{h_n}{h_n} = 1.$$

Now take $\omega_n = z_0 + ih_n$, where $\{h_n\} \subset \mathbb{R}$ is a real sequence converging to 0. Then we have

$$f(\omega_n) - f(z_0) = \overline{\omega_n} - \overline{z_0} = z_0 + \overline{ih_n} - \overline{z_0} = -ih_n \textit{ and } \omega_n - z_0 = ih_n,$$

which implies that

$$\frac{f(\omega_n) - f(\omega_0)}{\omega_n - \omega_0} = \frac{-ih_n}{ih_n} = -1.$$

Now we have two sequences, both converging to z_0 but give different limits. Then by Lemma 2.3, $f(z) = \overline{z}$ is not complex differentiable at any point in $\mathbb{C}$.

Definition 2.4. A function $f : \Omega \to \mathbb{C}$, with $\Omega \in \mathbb{C}$ open, is *holomorphic* if for all $z \in \Omega$, the complex derivative

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. We say that f is *holomorphic at the point* z_0 if f is complex differentiable on some neighborhood of z_0.

Example 2.3. All polynomial functions $f(z) = \sum_{i=0}^{n} c_i z^i$ with complex coefficients are holomorphic on $\mathbb{C}$.

Nonexample 2.3. $f(z) = \overline{z}$ is not holomorphic on $\mathbb{C}$.

2.2 Complex Charts and Complex Structures

2.2.1 Complex Charts

Definition 2.5. A *complex chart*, or simply a *chart*, on X is a homeomorphism $\phi : U \to V$, where $U \subset X$ is an open set in X, and $V \subset \mathbb{C}$ is an open set in the complex plane. The open set U is called the *domain* of the chart ϕ. The chart ϕ is said to be centered at $p \in U$ if $\phi(p) = 0$.

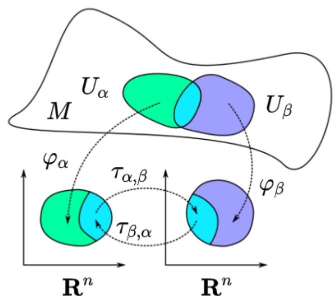

Figure 1. Two charts on a manifold and their respective transition map

Example 2.4. Let $X = \mathbb{R}^2$, and let U be any open subset. Define $\phi_U(x, y) = x + iy$ from U to the complex plane. This is a complex chart on $\mathbb{R}^2$.

Example 2.5. Again let $X = \mathbb{R}^2$. For any open subset U, define

$$\phi_U(x, y) = \frac{x}{1 + \sqrt{x^2 + y^2}} + i\frac{y}{1 + \sqrt{x^2 + y^2}},$$

this is also a complex chart on $\mathbb{R}^2$.

Example 2.6. Let $\phi : U \to V$ be a complex chart on X. Suppose that $U_1 \subset U$ is an open subset of U. Then $\phi|_{U_1} : U_1 \to \phi(U_1)$ is a complex chart on X. This restriction of ϕ is called a *sub-chart* of ϕ.

Example 2.7. Let $\phi : U \to V$ be a complex chart on X. Suppose that $\psi : V \to W$ is a holomorphic bijection between two open sets of the complex plane. Then the composition $\psi \circ \phi : U \to W$ is a complex chart on X. If we think of ϕ as a given complex chart on U, we can view this operation as a change of coordinates. The difference between the charts ϕ and $\psi \circ \phi$ is just simple change of coordinates, which does not change the entire structure on the open set.

Therefore, the "difference" between those two charts lead us to the following definition.

Definition 2.6. Let $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ be two complex charts on X. We say that ϕ_1 and ϕ_2 are *compatible* if either $U_1 \cap U_2 = \varnothing$, or

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is holomorphic. The function $T = \phi_2 \circ \phi_1^{-1}$ is called the *transition function* between the two charts and is always bijective.

This definition is symmetric: if $\phi_2 \circ \phi_1^{-1}$ is holomorphic on $\phi_1(U_1 \cap U_2)$, then $\phi_1 \circ \phi_2^{-1}$ is holomorphic on $\phi_2(U_1 \cap U_2)$.

Example 2.8. As we have mentioned in the previous example, let $\phi : U \to V$ be a complex chart on X, and let $\psi : V \to W$ be a holomorphic bijection between two open sets of the complex plane. Then the charts $\psi \circ \phi$ and ϕ are compatible. Moreover, $\psi \circ \phi$ will be compatible with any chart which is compatible with ϕ.

Example 2.9. Let S^2 denote the unit 2-sphere inside $\mathbb{R}^3$, i.e.,

$$S^2 = \{(x, y, w) \in \mathbb{R}^3 : x^2 + y^2 + w^2 = 1\}.$$

Consider the $w = 0$ plane as a copy of the complex plane $\mathbb{C}$, with $(x, y, 0)$ being matched with $z = x + iy$. Let $\phi_1 : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{C}$ be defined by projection from $(0, 0, 1)$. Specifically,

$$\phi_1(x, y, w) = \frac{x}{1 - w} + i\frac{y}{1 - w}.$$

The inverse to ϕ_1 is

$$\phi_1^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2 + 1}, \frac{2\,\mathrm{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right).$$

Define $\phi_2 : S^2 \setminus \{(0, 0, -1)\} \to \mathbb{C}$ by projection from $(0, 0, -1)$ followed by a complex conjugation

$$\phi_2(x, y, w) = \frac{x}{1 + w} - i\frac{y}{1 + w}.$$

The inverse to ϕ_2 is

$$\phi_2^{-1}(z) = (\frac{2\,\mathrm{Re}(z)}{|z|^2 + 1}, \frac{-2\,\mathrm{Im}(z)}{|z|^2 + 1}, \frac{1 - |z|^2}{|z|^2 + 1}).$$

The common domain is $S^2 \setminus \{(0, 0, \pm 1)\}$, and is mapped by both ϕ_1 and ϕ_2 bijectively onto $\mathbb{C}^* = \mathbb{C} - \{0\}$. The composition $\phi_2 \circ \phi_1^{-1}(z) = 1/z$, which is holomorphic on $\mathbb{C}^\infty$. Thus the two charts are compatible.

2.2.2 Complex Atlases

For X to look locally like the complex plane everywhere, we must have complex charts around every point of X and we want these charts to be compatible. This is the notion of a *complex atlas*.

Definition 2.7. A *complex atlas* (or simply *atlas*) $\mathcal{A}$ on X is a collection of pairwise compatible complex charts $\mathcal{A} = \{U_\alpha, \phi_\alpha\}$ for which the U_α constitute an open covering of M.

Remark 2.8. A point $p \in U_\alpha$ is uniquely determined by $\phi_\alpha(p)$. We may even omit the index α and call the components of $\phi(p) \in \mathbb{C}$ the *coordinates of* p.

The charts we defined in Example 2.4 form a complex atlas on $\mathbb{R}^2$, as do the charts in Example 2.5.

Example 2.10. If $\mathcal{A} = \{U_\alpha, f_\alpha\}$ is an atlas on X, and $Y \subset X$ is any open subset, then the collection of sub-charts $\mathcal{A}_Y = \{Y \cap U_\alpha, \phi_\alpha|_{Y \cap U_\alpha}\}$ is an atlas on Y.

Given two different atlases, when every chart of one atlas is compatible with every chart of the other atlas, those two atlases can give the same local notions of complex analysis on a Riemann surface. Therefore, we have an equivalence relation on atlases.

Definition 2.9. Two complex atlases $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if every chart of one is compatible with every chart of the other.

Note that two complex atlases are equivalent if and only if their union is also a complex atlas.

Definition 2.10. A *complex structure* on X is a maximal complex atlas on X, or, equivalently, an equivalence class of complex atlases on X.

Note that any atlas on X determines a unique complex structure. So, the usual way we define complex structures is by giving an atlas.

2.2.3 Manifolds and Differential Manifolds

Definition 2.11. A *manifold of dimension* n, or n*-manifold* is a connected Hausdorff space M such that for every point $p \in M$, there is an open set $U \subset S$ containing p, and a homeomorphism $\phi : U \to V$ onto an open subset $V \subset \mathbb{R}^n$. Such a homeomorphism

$$\phi : U \to V$$

is an n*-dimensional real chart on* X.

Remark 2.12.

- A point $p \in U_\alpha$ is uniquely determined by $f_\alpha(p)$. We may even omit the index α and call the components of $f(p) \in \mathbb{R}^n$ the coordinates of p.
- A manifold of dimension 2 is usually a surface.

We want to introduce a general tool for producing manifolds. The tool is the *Implicit Function Theorem*, a classic result from multivariable calculus. The complete version of Implicit Function Theorem and its proof can be found in almost any book on advanced calculus. For example, see [Spi65]. We will prove a special case below, a case that is fairly easy to prove yet still produces nice examples.
Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a continuous function. Assume also that the partial derivatives of f exist and are continuous functions. This means that the gradient ∇f exists and is continuous. We say that 0 is a *regular value* for f if it never happens that both $f(x_1, \cdots, x_{n+1}) = 0$ and $\nabla f(x_1, \cdots, x_{n+1}) = 0$ at the same point.

Proposition 2.13. If 0 is a regular value for f, then $f^{-1}(0)$ is an n-dimensional manifold.

Proof. See [Sch11] for a proof. ■

Example 2.11. Now we give a nice example of a 3-dimensional manifold. We can think of the set of 2×2 (real valued) matrices as a copy of $\mathbb{R}^4$. There is a nice map from this space into R, namely the determinant (minus 1):

$$f\left(\begin{vmatrix} a & b \\ c & d \end{vmatrix}\right) = ad - bc - 1.$$

In the above example, $f^{-1}(0)$ is usually denoted by $SL_2(\mathbb{R})$, which is the set of unit determinant real 2×2 matrices. By proposition 2.13, the space $SL_2(\mathbb{R})$ is a 3-dimensional manifold. By the similar argument, we can conclude that $SL_n(\mathbb{R})$, the set of unit determinant $n \times n$ matrices, is a manifold of dimension $n^2 - 1$.

Definition 2.14. An atlas $\{U_\alpha, \phi_\alpha\}$ on a manifold is called *differentiable* if for any two charts ϕ_1 and ϕ_2, either the intersection of their domain is empty, or

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to (\phi_2 U_1 \cap U_2)$$

is C^∞, i.e., has derivatives of all orders in its domain.

A chart is called *compatible* with a differentiable atlas if adding this chart to the atlas yields again a differentiable atlas. Taking all charts compatible with a given differentiable atlas yields a *differentiable structure*. A *differentiable manifold of dimension* d is a manifold of dimension d together with a differentiable structure.

Definition 2.15. A continuous map $h : M \to M'$ between differentiable manifolds M and M' with charts $\{U_\alpha, \phi_\alpha\}$ and $\{U'_\alpha, \phi'_\alpha\}$ is said to be *differentiable* if all the maps $\phi'_\beta \circ h \circ \phi_\alpha^{-1}$ are differentiable (of class C^∞) wherever they are defined.

Definition 2.16. If h is a homeomorphism and if both h and h^{-1} are differentiable, then h is called a *diffeomorphism*.

2.2.4 The Definition of a Riemann Surface

Definition 2.17. A topological space X is a *Hausdorff space* if for any two distinct points $x, y \in X$, there exist a neighborhood U of x and V of y such that $U \cap V = \varnothing$.

Example 2.12. The real numbers, under the standard metric topology, are a Hausdorff space.

Nonexample 2.4. A *cofinite* subset of a set X is a subset A whose complement in X is a finite set. The *cofinite topology* defined on a set X has precisely the empty set and all cofinite subsets of X as open sets, which can be written as

$$\mathscr{T} = \{A \subseteq X \mid A = \varnothing \text{ or } X \setminus A \text{ is finite}\}.$$

The cofinite topology defined on an infinite set is not Hausdorff.

Definition 2.18. A topological space X is *second countable* if there exists some countable collection $\mathcal{U} = \{U_i\}_{i=1}^\infty$ of open subsets of X such that any open subset of X can be written as a union of elements of some subset of $\mathcal{U}$.

Definition 2.19. A *Riemann surface* is a second countable, connected, and Hausdorff topological space X together with a complex structure.

The second countability condition is meant to exclude any pathological examples. Most examples we find naturally, like a subset of $\mathbb{C}^n$, are second countable. In particular, if the complex structure may be defined by a complex atlas, then X must be second countable.

Example 2.13. Let $X = \mathbb{C}$, considered topologically as $\mathbb{R}^2$, with the complex structure induced by the atlas of Example 2.4. This Riemann surface is called the *complex plane*.

Example 2.14. Let X be the 2-sphere, with complex structure given by the two-chart atlas of Example 2.9. This Riemann surface is called a *Riemann sphere*. Riemann sphere is often written as $\mathbb{C} \cap \infty$ or $\mathbb{C}_\infty$, with the complex plane $\mathbb{C}$ representing one chart, with the "point at infinity" ∞ being the single extra point. The Riemann sphere is a compact Riemann surface.

2.3 Functions on Riemann Surfaces

Let X be a Riemann surface, p a point of X, and f a function on X defined near p. To check whether f has any particular property at p (for example, to check whether f is holomorphic at p), we can use complex charts to transport the function to the neighborhood of a point in the complex plane, and check properties there. In this section we specify the checking process of various properties.
The only thing we need to be careful of is that the property we are checking must be independent of coordinate changes, so that it does not matter what chart we use to check the property.

2.3.1 Holomorphic Functions

Let X be a Riemann surface, let p be a point of X, and let f be a complex-valued function defined in a neighborhood W of p.

Definition 2.20. We say that f is *holomorphic* at p if there exists a chart $\phi : U \to V$ with $p \in U$, such that the composition $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$. We say f is *holomorphic in* W if it is holomorphic at every point in W.

We then have the following lemmas:

Lemma 2.21. Let X be a Riemann surface, let p be a point of X, and let f be a complex-valued function defined in a neighborhood W of p. Then:

(1) f is holomorphic at p if and only if for every chart $\phi : U \to V$ with $p \in U$, the composition $f \circ \phi^{-1}$ is holomorphic at $\phi(p)$.
(2) f is holomorphic in W if and only if there exists a set of charts $\{\phi_i : U_i \to V_i\}$ with $W \subseteq \bigcup_i U_i$ such that $f \circ \phi_i^{-1}$ is holomorphic on $\phi_i(W \cup U_i)$ for each i.
(3) if f is holomorphic at p, then f is holomorphic in a neighborhood of p.

Proof. See [Mir53] for a proof. ■

Definition 2.22. If $W \subset X$ is an open subset of a Riemann surface X, we will denote the set of holomorphic functions on W by $\mathcal{O}_X(W)$ (or simply $\mathcal{O}(W)$):

$$\mathcal{O}_X(W) = \mathcal{O}(W) = \{f : W \to \mathbb{C} \mid f \text{ is holomorphic }\}.$$

We note that $\mathcal{O}(W)$ is a $\mathbb{C}$-algebra.

2.4 More on Surfaces

2.4.1 Orientability

For any point on a smooth compact surface, there are two possible directions of rotation around this point. If such a direction is chosen at each point, and these directions are compatible for nearby points, then one says that the surface is endowed with an *orientation*. A surface S will be called *orientable* if we can define an orientation in a consistent way.

Example 2.15. Most surfaces we encounter in physical world are orientable. For example, we can define orientation on spheres, tori, and planes.

Example 2.16. The Klein bottle and the Mőbius strip are non-orientable.

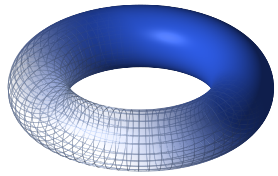

Figure 2. Orientable Surface

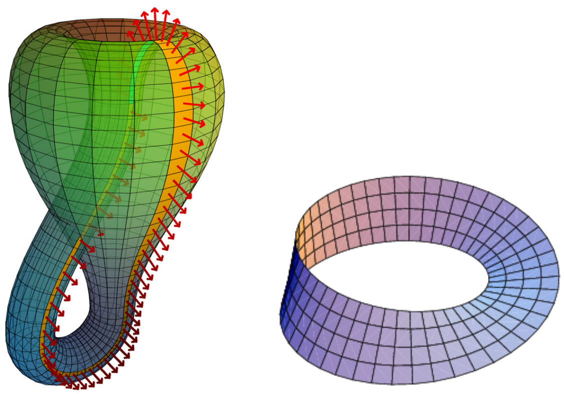

Figure 3. Non-orientable Surfaces

Remark 2.23. An intuitive way to determine whether a surface S is orientable is by detecting the existence of an *orientation-reversing curve*. This is a continuous curve that starts and ends at a single point $p \in S$, traversing a path $\gamma \subseteq S$ that includes points other than p. Then we choose a direction to move from p along γ and assign a right-handed orientation to each points on γ. When we come back to p, if we find that the y-axis defined by the right-handed orientation has changed direction, then γ is an orientation reversing curve.

In this paper, we will be interested in orientable surfaces only, because, from the topological point of view, a smooth complex curve is a closed orientable surface.

2.4.2 The Connected Sum

Another interesting operation of surfaces is the connected sum.

Definition 2.24. The *connected sum* of two surfaces M and N is the surface $M\#N$ obtained by removing small open disks D_1 and D_2 from M and N and gluing the surfaces $M \setminus D_1$ and $N \setminus D_2$ together by a homeomorphism $h : \partial D_1 \to \partial D_2$ (∂D denotes the boundary of D). What remains after removing a small open disk D from the torus $S_1 \times S_1$ is called a *handle*.

Example 2.17. The connected sum of a sphere and a torus is the same as attaching a cylinder at both boundary circles after removing two open disks, which is again a torus.

A *genus-g surface* (also known as a *g-torus* or *g-holed torus*) is a surface formed by the connected sum of g many tori, which is also known as a *sphere with g-handles*. Every closed orientable two-dimensional surface is homeomorphic to a sphere with g-handles (for some g), and spheres with different numbers of handles are not homeomorphic.

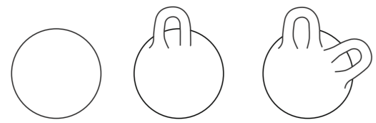

Figure 4. Handles

2.4.3 Triangulation and Euler Characteristic

Definition 2.25. Let X be a compact Riemann surface. A *triangulation* T of X is a collection of continuous maps,$t_i : T_i \to X$ for T_i from triangles in $\mathbb{C}$ onto X so that:

(1) the t_i are homeomorphisms onto their image
(2) the union of the image of all t_i cover X
(3) the intersection of any two triangles is either an edge or a point.

Every closed two-dimensional surface can be triangulated, i.e., cut into triangles in such a way that any two triangles either have no common points, or have one common vertex, or have one common side (but are not allowed to share only a part of a side). Given a triangulation of a surface M, let V be the number of vertices, E be the number of edges (sides), and F be the number of faces in this triangulation.

Definition 2.26. Let S be a compact 2-manifold, possibly with boundary. Suppose a triangulation is give with n vertices, e edges, and f faces. The *Euler characteristic* of S is

$$\chi(S) = v - e + f.$$

The main fact about Euler characteristic is that it does not depend on the particular triangulation one use to compute them.

Proposition 2.27. The Euler characteristic is independent of the choice of triangulation. For a compact orientable 2-manifold with boundary of topological genus g, its Euler characteristic is $2 - 2g$.

See [Mir53] for a proof.

2.5 Coverings

Definition 2.28. Let X be a topological space. A *covering space* of X is a topological space C together with a continuous surjective map $p : C \to X$ such that for every $x \in X$, there is an open neighborhood U of x, such that $p^{-1}(U)$ is a union of disjoint open sets in C, each of which is mapped homeomorphically onto U by the p. The map p is called a *covering map*, the space X is often called the *base space* of the covering, and the space C is called the *total space* of the covering.

Example 2.18. S^1 is the punctured unit disk $|z| < 1, z \neq 0$ in $\mathbb{C}$. The map $p : \mathbb{R} \to S^1$ defined by $p(t) = e^{it}$ is a covering map, wrapping the real line round and round the circle. For an little open arc in the circle, its preimage is a collection of open intervals in $\mathbb{R}$ with period of 2π.

Example 2.19. For any positive integer n, the mapping $p : S^1 \to S^1$ defined by $p(z) = z^n$ is a covering map. This wraps the circle around itself n times.

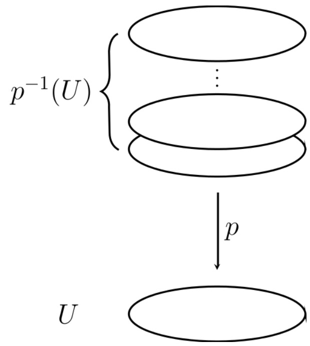

Figure 5. A Covering Space

Given a neighborhood U_x in the base space X, each point in U_x has the same number of preimages. Therefore, if the base space X is connected, the number of preimages of each point is constant over the whole space. This common number of preimages is called the *degree*, or the *number of sheets*, of the covering.

Theorem 2.29. *Let $p : C \to X$ be a n-sheeted covering with C and X being connected compact two-dimensional surfaces. Then $\chi(M) = n\chi(N)$.*

Proof. Consider a triangulation of N. Then the preimage of every triangle of N consists of n pairwise disjoint triangles, and all these triangles together form a triangulation of M. Since p is an n-sheeted covering, the latter contains exactly n triangles corresponding to each triangle of the original triangulation of N, exactly n edges corresponding to each edge, and exactly n vertices corresponding to each vertex. Therefore, $\chi(M) = n\chi(N)$. ■

If the covered surface N is orientable and one of the two orientations of N is chosen, then the covering surface M is also orientable, and it can be endowed with the orientation *induced* from N.

Definition 2.30. A *ramified covering* is a non-trivial holomorphic map between compact Riemann surfaces. Let $f : X \to Y$ be a ramified covering. A point P on X is *ramified* (or a *critical point*) if the differential of f at P vanishes. The *ramification index* is defined by $e_P = \min\{n \geq 1 \mid f^{(n)}(P) \neq 0\}$. If every point is not ramified, we call f an *unramified covering*.

Example 2.20. $P : \mathbb{C} \to \mathbb{C}^n$ defined by $P(z) = z^n$ is a ramified covering with critical points 0 and ∞, ramification index are n for both.

Example 2.21. $P : \mathbb{C} \to \mathbb{C}^4$ defined by $P(z) = z^3(z-1)$ is a ramified covering with critical points $0, \frac{3}{4}$, and ∞. Ramification index are 3, 1, and 4 respectively.

3 Riemann-Hurwitz Formula

As we have introduced in Theorem 2.29, for a covering, the Euler characteristic of the covering surface is given by a simple formula in terms of the Euler characteristic of the covered surface and the degree of the covering. Analogously, we want to see if we can have some similar formula when we study ramified coverings. And the answer turns out to be yes! In the case of ramified coverings, however, the formula turns out to be more complicated, involving some additional characteristics of the covering.

Theorem 3.1 (Riemann-Hurwitz). *Let $p : M \to N$ be an n-sheeted ramified covering with k ramification points having $m_1, \ldots, m_k$ preimages, respectively. Then*

$$\chi(M) = n(\chi(N) - k) + m_1 + \cdots + m_k.$$

Proof. Divide the surface N into two closed sets: $N = N_A \cup N_B$, where N_A is the union of the closures of small circular neighborhoods of the ramification points and N_B is the closure of the complement $N \setminus N_A$ to N_A. Then

$$\chi(N) = \chi(N_A) + \chi(N_B) - \chi(N_A \cap N_B).$$

But the set $N_A \cap N_B$ consists of several circles, hence $\chi(N_A \cap N_B) = 0$. Therefore,

$$\chi(N) = \chi(N_A) + \chi(N_B).$$

Similarly, we divide the surface M into the closed sets $M_A = p^{-1}(N_A)$ and $M_B = p^{-1}(N_B)$. Then

$$\chi(M) = \chi(M_A) + \chi(M_B).$$

The restriction of p to M_B is a covering, hence $\chi(M_B) = n\chi(N_B)$, and then

$$\chi(M) - \chi(M_A) = n(\chi(N) - \chi(N_A)).$$

Since the set M_A consists of $m_1 + m_2 + \cdots + m_k$ disjoint disks, and the Euler characteristic of a disk is 1, we have $\chi(M_A) = m_1 + \cdots + m_k$ and $\chi(N_A) = k$. ■

Here is another proof for Riemann-Hurwitz Theorem.

Proof. Consider a sufficiently fine triangulation of the surface N whose vertices include all ramification points. Here the words "sufficiently fine" mean that the preimage of the interior of each triangle of the triangulation consists of interiors of n triangles, where n is the degree of the covering, and the same holds for the preimages of interiors of edges. Over the interior of each triangle of the triangulation, as well as over the interior of each its edge, p is an unramified covering. Hence the p-preimages of the edges and the triangles of the triangulation of N form the induced triangulation of M. Let v_N, e_N, f_Nbe the number of vertices, edges, and faces of the triangulation of N. Let v_M, e_M, f_Mbe the number of vertices, edges, and faces of the induced triangulation of M. Then $e_M = ne_N$, $f_M = nf_N$, and $v_M = n(v_N - k) + \sum m_i$. Therefore,

$$\chi(M) = v_M - e_M + f_M = n\chi(N) - nk + \sum m_i.$$

■

Riemann-Hurwitz Formula shows us the relationship of the Euler characteristics of two surfaces when one is a ramified covering of the other. It turns out that we have an analogous formula in case of algebraic curves, which also has many applications, such as proving Lúroth's Theorem and Fermat's Last Theorem for Polynomials (see [Oor16]).

References

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