REAL PROJECTIVE GEOMETRY

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CONTENTS

1. MOTIVATION

This paper will look at projective geometry in the context of algebra.

To give a simple example of why study projective geometry, some situations are easier than in Euclidean geometry. For example 2 lines always intersect at one point, except when they don't. In projective geometry this degenerate case is absent.

I will present the fundamentals of projective geometry, aimed at being accessible to those with some background in abstract algebra. We will first build up some intuition for the algebraic side of projective geometry. This knowledge will help us work towards a special look at transformations in projective geometry.

2. INTRODUCTION

I will have a basic look at projective geometry, and see how projection works as opposed to Euclidean Geometry

3. AXIOMS OF PROJECTIVE GEOMETRY

Each mathematical system relies on its foundational axioms. In this paper I will use Whitehead's axioms, which are quite simple.

Definition 1. *G1: Every line contains at least 3 points G2: Every two distinct points, A and B, lie on a unique line, AB. G3: If lines AB and CD intersect, then so do lines AC and BD*

It is more important what is not included, namely that of parallel lines, which is in Euclidean Geometry.

4. ALGEBRAIC DEFINITIONS

Definition 2. *Let's first start out with $\mathbb{R}^{n+1}$, which is an n+1 dimensional vector space, or just and ordered tuple $(a_1, a_2 ... a_{n+1})$ with $a_i \in \mathbb{R}$. $\mathbb{R}^{n+1}$ corresponds to $\mathbb{P}^n$.*

You should imagine points in affine space corresponding to lines in projective space, with the line passing through that point and the origin. This allows us to see why at least one of the coordinates of a projective point must be nonzero.

An important concept in projective geometry is that of homogeneity.

Definition 3. *2 points X, Y are homogeneous if for every coordinate x and y, $\exists$ k s.t.* $x_i = ky_i$

This notion is similar to how 2 points are on the same line if their coordinates are in a ratio. In fact, the way we write projective coordinates, $(p_0 : p_1 ... p_n)$ with colons represents how these coordinates are in a ratio.

5. Point at Infinity

Before, in the introduction, I mentioned that 2 lines always intersect in projective geometry, but how does this happen? This involves the point at infinity. All lines intersect at exactly 1 point. If 2 lines have the same slope, then they have a unique point at infinity that they intersect at.

6. Homography

One of the key transformations in projective space is the homography.

Definition 4. *Given 2 Productive spaces, $\mathbb{P}_a$ and $\mathbb{P}_b$, which are both of the same dimension then a homography is a mapping induced by a bijection between between the 2 vector spaces they are based on.*

A simple example of a homography is $z \to \frac{az+b}{cz+d}$ when $ad - bc \neq 0$. This is called a line homography.

7. First look at projective transformations

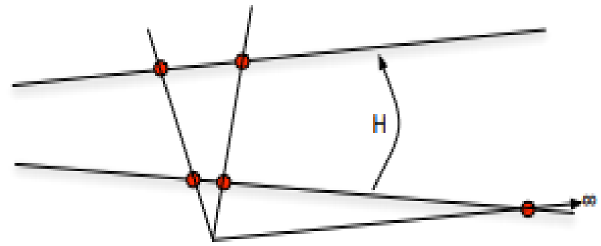
We will first look over $\mathbb{P}^1$ because that is easiest to conceptualize.

We will represent a transformation as a matrix $\begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}$ When we represent transformations as matrices, we are applying matrix multiplication to each point in the projective space. As a simple example, let the point p = $\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$ and the transformation will be $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 2 \\ 1 & 0 & 1. \end{bmatrix}$ Then this point will be mapped to $\begin{bmatrix} 1 \\ 3+2i \\ 1 \end{bmatrix}$

In any transformation, we should look at the invariants, or what is not changed by the transformation. For projective transformations, that is the cross-ratio.

Definition 5. *The cross ratio between 4 points, A,B, C, and D, is defined as $\frac{|AC||BD|}{|BC||AD|}$, where* $|AB| = (x_0^A x_1^B) - (x_1^A x_0^B)$

In fact, the cross ratio is so important to projective geometry that Mundy writes, "It seems likely that all invariant properties of a geometric configuration can ultimately be interpreted in terms of some number of cross-ratio constructions."



To see a visual representation of a projective transformation, the graphic to the right demonstrates that 3 correspondences are enough to determine a transformation.

8. Proof of Projective Transformations

It can be good to verify some of the properties of projective transformations with a proof.

For instance, let's say we have a projective transformations, A, defined by a 3x3 matrix.

We have [x : y : 1]A = [x': y': z'] with z' = $a_{1,3}x + a_{2,3}y + a_{3,3}$. If A induces an affine transformation, then z' $\neq$ 0 for all x, y $\in$ K. Since det(A) can never be 0, $a_{2,3} = a_{3,3} = 0$.

If $a_{1,3} = a_{2,3} = 0$, then $A([x : y : 0]) = [x' : y' :']$, hence the line z = 0 is preserved.

9. Affine Transformations

Another type of transformation we might want to look at in projective space is the affine transformation.

Here is an example of the matrix of an affine transformation $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$

They are actually a subset of the projective transformations.

Affine transformations preserve parallels, which means any lines which intersect at infinity still do after an affine transformation.

10. Isometry

The final type of transformation I will look at is the isometry. Let's first start out with a definition

Definition 6. *An isometry is a transformation which preserves distance.*

Isometries are of the form
$\begin{bmatrix} x\cos\theta & x\sin\theta & a_0 \\ \sin\theta & \cos\theta & a_1 \\ 0 & 0 & 1 \end{bmatrix}$ where $x = \pm 1$.

These are probably the most straight-forward transformations, in my opinion, because they preserve too much of the pre-image.

11. Comparison

I will make a table now comparing each of the transformations

name	matrix	invariants
projective	$\begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$	cross-ratio
affine	$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$	slope
isometry	$\begin{bmatrix} x\cos\theta & x\sin\theta & a_0 \\ \sin\theta & \cos\theta & a_1 \\ 0 & 0 & 1 \end{bmatrix}$	distance

12. Further Reading

I suggest some papers and books that dive deeper into Real Projective Geometry.

I suggest 'Projective Geometry: An Introduction' by Rey Casse or 'The Real Projective Plane' by Coxeter

References

[1] Reinhold Baer. *Linear algebra and projective geometry*. Courier Corporation, 2005.

[2] Harold Scott Macdonald Coxeter. *Projective geometry*. Springer Science & Business Media, 2003.

[3] Oswald Veblen and John Wesley Young. *Projective geometry*, volume 2. Blaisdell, 1918.

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