Klein's Quartic

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1 Introduction

In November 1933, a sculpture was unveiled in the Mathematical Science Research Institute (MSRI), Berkley. It was a commission from the mathematical sculptor Ferguson. The sculpture is a three dimensional visual of the Klein Quartic, known as the Eightfold Way. The sculpture depicts the beautifully unique surface discovered by Felix Klein in 1878, whose equation is simply given as $X^3Y + Y^3Z + Z^3X = 0$.

In this paper, we will look at some of the characteristics of the Klein Quartic, which will be denoted as $\mathcal{X}$, such as its construction, genus and automorphisms. An in depth construction of $\mathcal{X}$ will be established through the use of hyperbolic planes and tessellations. Then, we will begin to look at the quartic's derivation as a Riemann Surface, before looking into its properties and uniqueness as a Hurwitz curve.

2 Preliminary Details

The following terms are important to understand, before studying $\mathcal{X}$.

Definition 2.1. A Riemann surface is a surface that covers the complex plane with several "sheets" that can have complicated structures and interconnections. In more simpler terms, a Riemann surface is a surface that represents the domain of complex functions.

Definition 2.2. An automorphism is an isomorphism from one group to itself. In the simplest sense when talking in context of a surface such as $\mathcal{X}$ an automorphism can be considered as the symmetry of the surface.

Definition 2.3. The automorphism group of an algebraic object O, denoted $\mathrm{Aut}(O)$ is the group of automorphisms of O that preserve the structure of the object. So, in the case of $\mathcal{X}$, it would be referring to the group of orientation preserving symmetries.

Definition 2.4. A Reimann surface with maximal number of automorphisms, considered an algebraic curve over $\mathbb{C}$ is called a Hurwitz Curve.

Definition 2.5. The cyclic group of a surface is the set of rotational symmetries.

Definition 2.6. A platonic surface is a Riemann surface that underlies a regular map so that it vertices, edge-centers and face centers can be considered.

Definition 2.7. given a ring R, the special linear group, denoted $\mathrm{SL}(n, r)$ is the set of $n \times n$ matrices with elements in R and determinant 1.

An extension of this is the projective special linear group.

Definition 2.8. A Projective Space Linear Group the group of $n \times n$ matrices with determinant of 1, which are scalar multiples of q. It is denoted as $\mathrm{PSL}(n, q)$.

3 Basic Characteristics

Before we go any further it's important that we begin by looking at what the Klein Quartic actually is. The Klein Quartic is a compact Reimann Surface that can be viewed in $\mathbb{P}^2$ as the curve $x^3y = y^3z + z^3x = 0$. $\mathcal{X}$ has a genus 3 classified as automorphism group G, with 168 orientation preserving automorphisms , which is the maximum size of an automorphism group for any curve of genus 3 (a fact we will look deeper into when looking at Hurwitz curves). We will look into why the curve has genus 3 and so many automorphisms through a visualization and 3D construction of the curve using hyperbolic surfaces and tessellations.

4 Construction of the Curve Using Hyperbolic Tessellations

We can begin by looking at a tessellation by $\frac{2\pi}{3}$-heptagons. We can identify using Euler's formula the number F of faces, $V = \frac{7}{3}F$ of vertices and $E\frac{7}{2}F$ edges. This would then give us $F = 24, V = 56$, and $E = 84$. Now, if we consider the cyclic group of prime order p on a genus 3 surface that has f fixed points, then the Euler number for the quotient surface gives:

$$\frac{1}{p}((V - f) - E + F) + f = \frac{1}{p}(-4 - f) + f \in \{-2, 0, 2\}$$

or in a simpler form,

$$f \in \{-2 + \frac{2}{p-1}, \frac{4}{p-1}, 2 + \frac{6}{p-1}\} \cap \mathbb{Z}$$

Hence $p = 7$ is the maximal prime order, with $f = 3$ and the quotient being a sphere. Therefore, a genus 3 surface with an order 7 cyclic group of automorphisms has a quotient map to the sphere. We can identify the sphere with $\mathbb{C} \cup \{\infty\}$, and send the three fixed points to $0, 1, \infty$.

If we were to look at a surface with $p = 2$, or an involution, then we must have $f = 0, 4$ or 8 fixed points. It's important to note that an involution by $2\pi/3$-heptagons cannot have fixed points at vertices or centers of faces, meaning that the fixed points must be at edge midpoints. Then, is this case f must divide E or the number of edges, meaning that $f = 8$ cannot occur when $E = 84$, therefore showing that the quotient is not a sphere, meaning that $\mathcal{X}$ is not hyperelliptic.

Since the desired cyclic symmetry group is of order 7, we can start by arranging 14 triangles to form a $2\pi/7$-fourteengon as seen in figure 1. In order to provide a smooth hyperbolic surface, all of the even and odd vertices need to be identified, which leaves us with three possibilities: identify edge 1 through 4, 6, or 8. The last option has 180°-rotation around the center and $f = 8$ fixed points, meaning that the quotient is a sphere. Hence, this example is a hyperelliptic surface and not the solution we are looking for.

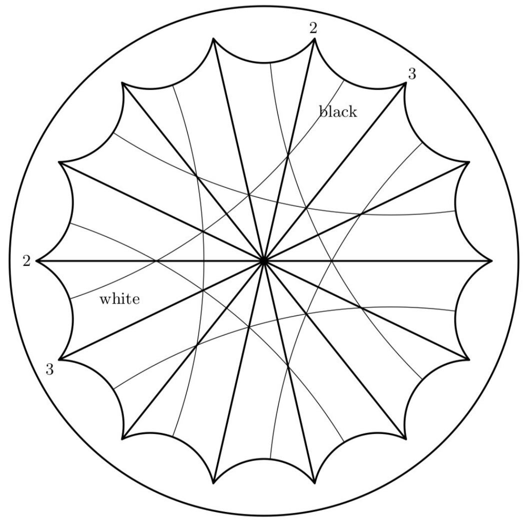

Figure 1: hyperbolic fourteengon

If we were to choose the option of identification of edge 1 to 4, we would have the same hyperelliptic surface, so that leaves the identification of edge 1 through 6 as the only option for a platonic surface, which we will show is $\mathcal{X}$. Now, if we start by coloring the fourteen triangles alternating black and white, each black edge can be identified by the white edge five counterclockwise steps ahead. Then, we can call the center vertex 1, the left endpoint of the black edge 2, and the right endpoint vertex 3. However as can be seen in figure 1, translation of a black edge to a white one is done by rotating vertex 2 by $2 * 2\pi/7$; and similarly, vertex 3 by $3 * 2\pi/7$ around the center. To express this in simpler terms,

$$2 * \{1, 2, 4\} = \{2, 4, 1\} \mod 7$$

and

$$4 * \{1, 2, 4\} = \{4, 1, 2\} \mod 7$$

By this identification rule, we can see that high symmetry in $\mathcal{X}$ is apparent in the fact that the rule remains the same (mod 7) if we cyclically permute the vertices. From here, we can see the formation of sphere that is thrice punctured.

Now, to apply the new identification rule we consider the tessellation of the hyperbolic plane by the black and white $\pi/7$ triangles. If we look at the equivalence classes of the triangles and vertices, we can see that the rule allows us to pick any arbitrary triangle from each class and know how to identify it. Additionally we also saw that a 120° rotation around any center that cyclically permutes the equivalence classes of the vertices, does not affect the identification rule. The same goes for the reflection in a triangle edge. Such reflections simply generate the order 7 symmetry and hence pass to the quotient sphere. This means that we can once again understand the quotient map through a Riemann mapping problem. We want to map a black triangle to the upper half plane, so that the vertices 3,2,1 go to $0, 1, \infty$ and extend by reflection.

We can define a function with the Riemann mapping theorem, bu mapping one of the black triangles to a spherical domain bounded by two circles from 0 to ∞ with the angle $3 * \pi/7$ at ∞. Additionally, there is a slit from 0 dividing the angle at 0 as $2 : 1$ with the bigger angle first counterclockwise. Then if we analytically extend this map using reflection in the edges to encompass the sphere three times, the slits in the three sheets are such that there is a slit in two sheets above each other, but not in the third. This form of forced identification of the slits is also compatible with the identification on the edges of the fourteengon since the rotational angles at the vertices of the black triangle are the same as the rotational angles at the vertices of the spherical domain. Then, if we compare the divisors of the function w and the quotient function z, we find that w^7 and $z(z-1)^2$ are proportional, and that we can scale w to give the the following equation:

$$w^7 = z(z-1)^2$$

From this we can define the famous quartic equation that exhibits the order 7 symmetry.

If we start by defining the following: $x := (1-z)/w^2 = -v^{-1}$, and $y := -(1-z)/w^3 = +u$.
The first equation then implies the quartic equation $w^3y + y^3 + x = 0$. However, the substitution can also be inverted as follows $w = -x/y$, $z = 1 - x^3/y^2$. Additionally we can also define the functions x, y as quotients of holomorphic 1-forms:

$$x = \xi/\omega, y = \eta/\omega$$

This then leads us to the equation in its homegeneous form

$$\xi^3\eta + \eta^3\omega + \omega^3\xi = 0$$

At this point, the heptagon tessellation can be introduced to our previous picture to easily prove platonicity. One $2\pi/3$ heptagon can be tessellated by fourteen $(2,3,7)$ - triangles and that fit around its center. We now want to tessellate one of the black triangles by 24 of the small triangles. If we take half of a heptagon that is tessellated by seven small triangles, to the left of the diameter and reflect the lowest $(2,3,7)$-triangle, we get eight small triangles. Since the are of the big triangle is equal to 24 of the smaller triangles, we have now tessellated one third of the big triangle. Now, if we do $120°$ rotations around the heptagon vertex 2, we can complete the tessellation of the big triangle. Then we can extend this by reflection to get Klein's tessellation of the hyperbolic plane. If we look closely, we can also notice that we can group the tessellation by either heptagons or big triangles.

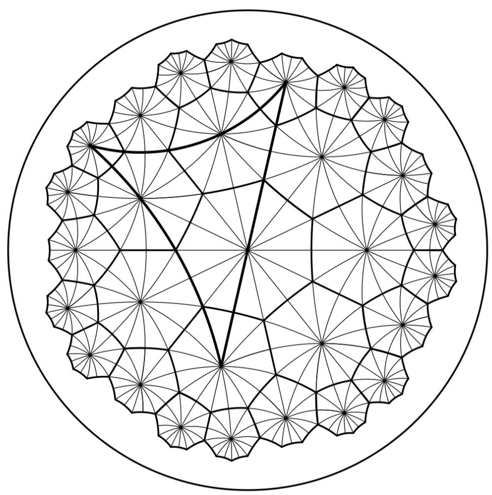

Figure 2: tessellation using heptagons/big triangles

As we can see above, we have now tessellated the above Riemann surface by 24 regular heptagons. The identification translations are compositions of involutions around midpoints of the edges of the heptagon's edges, which is another indication that the surface at hand is platonic.

We now need to check that all eight of the segment geodesics, or straight lines on the sphere, that connect midpoint of the edges of the heptagons also connect equivalent points under the deck group, since that is the last condition to be met in order for the Riemann surface to be platonic (the deck group is generated by the identification translations).If we look at the modulo reflections in the symmetry lines through the center, there are only four different eight segment geodesics that meet the fundamental domain. Then, with the 120°-rotations, these can be rotated into the ones that were used to define the identifications. Our hyperbolic definition is now complete enough to see platonicity since the 180°-rotations around the midpoints of the edges always sends equivalent points to equivalent points.

Now that we have a well defined tessellation of 24 regular heptagons, we can now warp this surface to form a three dimensional visualization of the quartic. If we refer to the below picture,we can see that that the tessellation has been number one through twenty four.

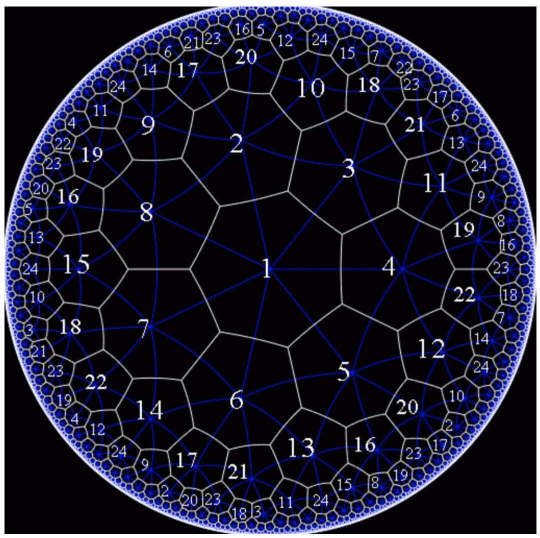

Figure 3: tessellation using heptagons/big triangles with numbers

Now, we can warp this into by making sure that all heptagons with same numbers are attached together. Another way of warping this tessellation, is by taking the edges of the 14 big triangles and connecting them in a specific order. To be exact, we want to connect edges $2n + 1$ and $2n + 16 \mod 14$ This will then result in a three hold torus that looks similar to the figure below.

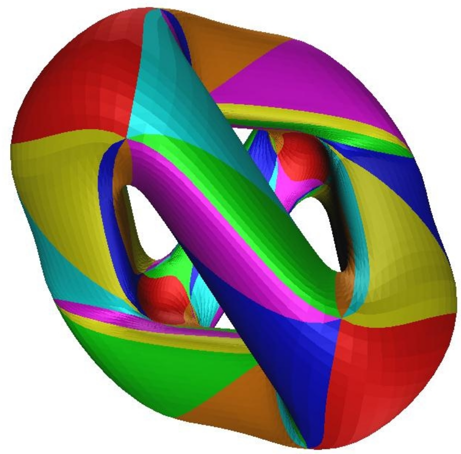

Figure 4: a visualization of the warped tessellation

When we look at the original tessellation, we can visualize the 24×7-fold symmetry of the quartic. This is what gives us 168 symmetries.

5 $\mathcal{X}$ as a Riemann Surface and Hurwitz Theorem

The Klein Quartic belongs to the infinite family PSL$(2, 7)$ and PSL$(3, 2)$. However, from how Klein studied it, the main focus is PSL$(2, 7)$. Klein's approach to studying this, the group and Riemann Surface, was by studying the group $\mathcal{T}(1)$ of all functions:

$$z \mapsto \frac{pz + q}{rz + s}$$

where $p, q, r, s \in \mathbb{Z}$, and $ps - qr = 1$, which are the permutations of the upper half plane $\mathbb{U} : +\{z \in \mathbb{C} | i(\bar{z} - z) > 0\}$ However,$\mathcal{T}(1)$ is a discontinuous group of maps, since the integers are a discrete subset of the real numbers. Hence, $\mathbb{U}$ is a Riemann surface, so its quotient, $\mathbb{U}/\mathcal{T}(1)$ is also a Riemann Surface. Specifically, the quotient surface is a sphere with a *puncture*, or one missing point. However, to find more interesting Riemann surfaces, we can also look at subgroups of $\mathcal{T}(1)$.

We can start by looking at the congruence subgroups.

Definition 5.1. The congruence subgroups $\mathcal{T}(n)$ consists of mappings such that

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \equiv \pm Id \mod n$$

The hidden treasure here, is $\mathcal{T}(7)$. It is a surface with genus 3, 24 punctures that are removable singularities, and 168 automorphisms. This quotient group is the result of replacing the integers in our initial mapping with their residue classes modulo 7, which we denoted $\mathrm{PSL}(2, 7)$. This group acts as the symmetries of Klein's quartic curve.

Now that we have a clearer idea of $\mathbb{X}$ as a Riemann surface, we can start looking at one of it's special characteristics as a Hurwitz curve.

Theorem 1. *A classical theorem of Hurwtiz states that if a Reimann Surface has a genus $g > 1$, then it can have at most $84(g-1)$ automorphisms (orientation preserving symmetries).*

Since when $g = 3$, the maximum value is 168, the Klein's quartic has a maximal automorphism group making it a Hurwitz curve. However, when looking at the theorem the appearance of the number 84 seems somewhat random, but, there is in fact a reason why such a number appears in this fundamental result. It's because in order to get Riemann surfaces that are as symmetrical as possible, we would start with tiling the hyperbolic plane with triangles with vertex angles $\pi/p, \pi/q, \pi/r$ for $p, q, r \in \mathbb{Z}$. In the case of $\mathcal{X}$, we take $p = 2, q = 3, r = 7$. Now, if we want to tile a torus with genus g, Euler's formula states the following: $V - E + F = 2 - 2g$ (Recall that we used this earlier in the paper). However, $E = \frac{3}{2}F$ since each face has 3 edges, but 2 faces share each edge. We can also figure out the number of vertices base on the number of faces. In highly symmetric tilings, there are vertices where $2p$ triangles meet, $2q$ triangles meet and $2r$ triangles meet. Hence, that would give us $V = (\frac{1}{2p} + \frac{1}{2q} + \frac{1}{2r})F$. Then, by combining the previous three equations, we get

$$\frac{4(g-1)}{1 - (\frac{1}{2p} + \frac{1}{2q} + \frac{1}{2r})}$$

. In order to maximize this value, we need to minimize the denominator, which we can get by using $p = 2, q = 3, r = 7$. This would then give us a denominator of $\frac{1}{42}$. Then going back to our combined equation, we get a maximum of $42 \times 4(g-1) = 168(g-1)$. However, these symmetries include reflection which are usually excluded, as we stated in the theorem earlier. Hence, we have to divide our expression by two, giving us

$$84(g-1).$$

Hence, thid explains the beautiful symmetricity of the quartic and it's maximal automorphism group, according to the Hurwitz theorem.

6 Summary

Through this paper, we have developed a deeper understanding of what may initially seem as a simple curve. We looked at an in depth construction of the curve using hyperbolic surfaces and tessellations. We also examined $\mathcal{X}$ in the context of Riemann surfaces and explored a little bit about Hurwitz curves, using $\mathcal{X}$ as an example. Though we didn't explore these in our paper, the Klein Quartic also has many other purposes. For example, in the field of number theory, $\mathcal{X}$ and some properties of its automorphism group can be used in a proof of Fermat's theorem for exponent 7. Other areas of math where the quartic is used, includes representation theory and homology theory. It can also be used to prove Stark-Heegner theorem on imaginary quadratic number fields of class number 1.

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