Tropical Algebra

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1 Introduction

In this paper, I will be discussing tropical algebra: starting with the basic arithmetic, moving into polynomials, and eventually leading into an interesting optimization application.

2 The Basics

2.1 Arithmetic

The *Tropical Semiring*: $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

Basic operations:

$a \oplus b = min(a, b)$ and $a \odot b = a + b$

Example: $2 \oplus 5 = min(2, 5) = 2$, and $2 \odot 5 = 2 + 5 = 7$

2.2 Properties of the Arithmetic

Both tropical addition and multiplication are *commutative*, so it follows that

$a \oplus b = b \oplus a$ and $a \odot b = b \odot a$

This should be somewhat intuitive, because $min(a, b) = min(b, a)$ and $a+b = b + a$.

Similar to the order of operations in standard algebra, tropical multiplication always has to come before addition. For example, $3 \odot 5 \oplus 6 \neq 3 \odot (5 \oplus 6)$, because this would be $3 + min(5, 6) = 3 + 5 = 8$. In reality, $3 \odot 5 \oplus 6 = (3 \odot 5) \oplus 6$, or $min(3 + 5, 6) = min(8, 6) = 6$.

The *distributive* law holds true in tropical arithmetic (as an exercise, prove this!):

$a \odot (b \oplus c) = a \odot b \oplus a \odot c$.

Example: $5 \odot (2 \oplus 7) = 5 \odot 2 = 5 + 2 = 7$. This expression can also be distributed: $5 \odot 2 \oplus 5 \odot 7 = 7 \oplus 12 = 7$.

Both the addition and multiplication operations have an *identity element*, or *neutral element*: the identity element for addition is ∞ and the identity element for multiplication is 0. This means that

$x \oplus \infty = x$ and $x \odot 0 = x$

Because 0 is the multiplicative identity, not 1, then $x \neq 1 \odot x$, because $x \neq x+1$. This is just something to watch out for.

There is no subtraction in tropical algebra, because this can quickly lead to problems. For example, if x is 11 minus 8, this would mean that $x \oplus 8 = 11$. But because the minimum of 8 and x will always be at most 8, this can't be true.

As for exponents, it is easy to figure out that, $a^b = b^a = a \times b$ in tropical algebra.

In tropical algebra, the *"Freshman's Dream"*, which says that $(x+y)^n = x^n + y^n$ for $n \in \mathbb{N}$, actually holds true:

$(x \oplus y)^n = x^n \oplus y^n$.

This is because $(x \oplus y)^n = (min(x, y) * n)$, and because n is positive, then $min(x, y) * n = min(nx, ny) = x^n \oplus y^n$.

3 Use in Polynomials

Let $x_1, x_2, \ldots, x_n \in (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ A tropical monomial looks like this:

$x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$,

where each $x_i^{k_i} = \underbrace{x_i \odot x_i \odot \cdots \odot x_i}_{k_i times}$. This means that in classical arithmetic $x_i^{k_i} = \underbrace{x_i + x_i + \cdots + x_i}_{k_i times} = x_i * k_i$. It follows that

$x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} = (x_1 * k_1) \odot (x_2 \dot{k}_2) \odot \cdots \odot (x_n * k_n) = x_1 k_1 + x_2 k_2 + \cdots + x_n k_n$.

For example, $2^4 5^3 6^2 = 2(4) + 5(3) + 6(2) = 8 + 15 + 12 = 35$.

3.1 Definition 2.1:

A *tropical polynomial* is a finite linear combination of tropical monomials:

$p(x_1, \ldots, x_n) = a_1 \odot x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \oplus a_2 \odot x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n} \oplus \ldots$

In $p(x_1, \ldots, x_n)$, the coefficients $a_1, a_2, ... \in \mathbb{R}$ and the exponents $i_1, j_1, ... \in \mathbb{Z}$.

(**Note** : unlike in classical algebra, exponents in a tropical polynomial can be negative. This makes tropical polynomials *Laurent polynomials*, which can be expressed as elements of $K[x, \frac{1}{x}]$ for a field K.)

Written in classical arithmetic, the polynomial looks like this:

$p(x_1, \ldots, x_n) = min(a_1 + x_1 i_1 + \ldots x_n i_n, a_2 + x_1 j_1 + \ldots x_n j_n, \ldots)$

It is just a minimum of a finite number of linear equations.

The function $p : \mathbb{R}^n \to \mathbb{R}$ satisfies three important properties:

- p is continuous,
- p is piecewise-linear, where the number of pieces is finite,
- p is concave, meaning $p(\frac{x+y}{2}) \geq \frac{1}{2}(p(x) + p(y))$ for any $x, y \in \mathbb{R}^n$

3.2 Fact 2.2:

All tropical polynomials $p(x_1, \ldots, x_n)$ are the piecewise-linear concave functions on $\mathbb{R}^n$ with integer coefficients.

Example: Let $p(x) = x^2 \oplus 1$, then $p(x) = x \odot x \oplus 1$. Written in classical algebra, $p(x) = min(2x, 1)$. This would be a piecewise-linear function with two different lines: when $2x \geq 1$, or $x \geq \frac{1}{2}$, then $p(x) = 1$, and when $2x < 1$, or $x < \frac{1}{2}$, then $p(x) = 2x$. $p(x)$ can be written as a piecewise function:

$$p(x) = \begin{cases} 1 & x \geq \frac{1}{2} \\ 2x & x < \frac{1}{2} \end{cases}$$

In this case the real zero of $p(x)$ is when $x = 0$.

Example: Let us take the equation of a unit circle, so let $p(x, y) = x^2 \oplus y^2 \oplus (-1)$. Then $p(x) = min(2x, 2y, -1)$. Then there are no real zeros, because it would have to be that either $x = 0$ or $y = 0$, but then the minimum would just be -1. What if instead the 1 is positive? Let $r(x, y) = x^2 \oplus y^2 \oplus 1$. Then the zeros of r are where $x = 0 \leq y$ and where $y = 0 \leq x$. From this example we can observe that if there is a constant in a polynomial, the constant must be positive for it to have any real zeros.

Example: Let $p(x) = a \odot x^2 \oplus b \odot x \oplus c$ be a tropical quadratic in one variable. We can see that $p(x) = min(2x + a, x + b, c)$, where this will split the graph in the (x,y) plane into three lines: $y = 2x + a, y = x + b$, and $y = c$. It will only be true that $y = 2x + a$ when $2x + a \leq x + b$, meaning $x \leq b - a$, and when $2x + a \leq c$, meaning $c - a \geq 2x$. Similarly, $y = x + b$ only when $x \geq b - a$, and when $x \leq b - c$. Finally, $y = c$ only when $c - a \leq 2x$ and $c - b \leq x$.

For a polynomial in one variable $p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0$, all lines are present in the graph of $y = p(x)$ on the (x,y) plane if

$a_1 - a_0 \leq a_2 - a_1 \leq \cdots \leq a_{n-1} - a_{n-2} \leq a_n - a_{n-1}$.

Proof: Let $p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0$. This can be expressed as a minimum of a collection of lines: $p(x) = min(nx + a_n, (n-1)x + a_{n-1}, \ldots, x + a_1, a_0)$. If one of these lines $y = mx + a_m$ is less than or equal to in y-value at the intersection of lines $y = (m-1)x + a_{m-1}$ and $y = (m+1)x + a_{m+1}$, then it will be present in the graph. The point of intersection is where $y = (m-1)x + a_{m-1} = (m+1)x + a_{m+1}$. This returns $x = \frac{a_{m-1} - a_{m+1}}{2}$. The line $y = mx + a_m$ will only be present in the graph of $p(x)$ if $mx + a_m \leq (m+1)x + a_{m+1}$, which means when $a_m - a_{m+1} \leq x = \frac{a_{m-1} - a_{m+1}}{2}$. This leads to $a_m - a_{m-1} \leq a_{m+1} - a_m$ for any $0 < m < n$. Therefore, all lines are included in the graph if and only if $a_1 - a_0 \leq a_2 - a_1 \leq \cdots \leq a_{n-1} - a_{n-2} \leq a_n - a_{n-1}$.

The *fundamental theorem of algebra* holds: not necessarily every tropical polynomial can be factored into linear equations, but every tropical polynomial is equivalent to some other polynomial which can be factored into linear equations.

For example with the polynomial $p(x) = 2 \odot x^2 \oplus 4 \odot x \oplus 3$, this is equivalent to $q(x) = 2 \odot x^2 \oplus 3$. This is because the line $y = x + 4$ is always greater than $y = 2x + 2$ or $y = 3$, so it is not even a part of the picture here.

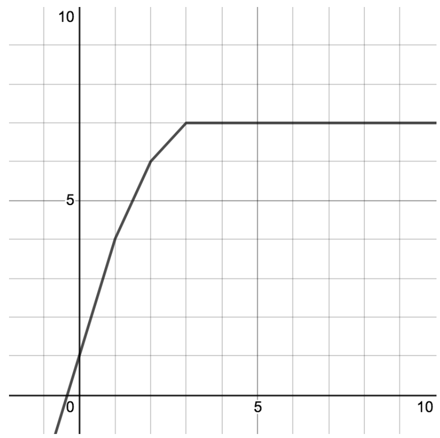

Figure 1: graph of $p(x) = 1 \odot x^3 \oplus 2 \odot x^2 \oplus 4 \odot x \oplus 7$

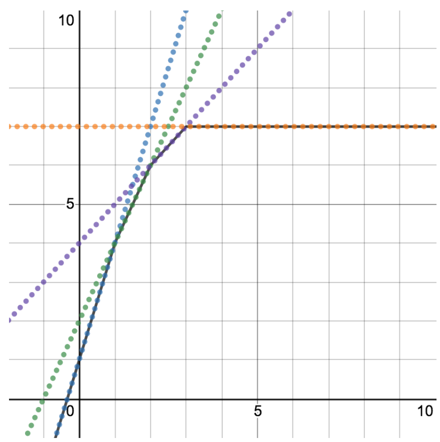

Figure 2: graph of $p(x) = 1 \odot x^3 \oplus 2 \odot x^2 \oplus 4 \odot x \oplus 7$ showing the different lines

$$4 \odot x^2 \oplus 2 \odot x \oplus 3 = (3 \odot x \oplus 1) \odot (1 \odot x \oplus 2).$$

3.3 Hypersurface:

For a tropical polynomial $p : \mathbb{R}^n \to \mathbb{R}$, the *hypersurface*, denoted by $\mathcal{H}(p)$, is the set of all points $x \in \mathbb{R}^n$ where p(x) is the minimum of two or more lines. A point x will be an element of $\mathcal{H}(p)$ if and only if $p(x)$ is not linear at x. The hypersurface $\mathcal{H}(p)$ can be thought of as the "roots" of $p(x)$.

3.4 Polynomials in Two Variables:

For any polynomial $p(x, y)$, $\mathcal{H}(p)$ is a tropical curve that is a finite graph in $\mathbb{R}^2$. All edges in the graph of $\mathcal{H}(p)$ will have rational slopes.

For any node (x, y) of the graph, take the smallest nonzero lattice vector of each line coming from this node. *Zero tension* at (x, y) means the sum of all these vectors is zero. For example if there was a line going vertically up from the node - this would correspond to the vector $(0, 1)$, a line going horizontally left from the node - corresponding to $(1, 0)$, and a line going down and left from the node - $(-1, -1)$, then there is zero tension at this node because $(1, 0) + (0, 1) + (-1, -1) = (0, 0)$.

For example with a line in two variables:

$p(x, y) = a \odot x \oplus b \odot y \oplus c$, where $a, b, c \in \mathbb{R}$

If would follow that $\mathcal{H}(p)$ consists of all points (x, y) such that $p(x, y)$ is equal to the minimum of at least two of the lines $a + x, b + y$, and c.

4 Optimization problems:

By now, you the reader have probably figured out that tropical algebra can be used for optimization. Optimization often involves a set of different lines and you want to optimize, or attain the most you can with the least effort possible, which very often involves a minimum!

Example:

You are taking a taxi to a location 6 miles away, and you need to go at least 3 miles in a taxi but can walk the rest of the way if necessary. You have \$12 on you. Taxi A charges a flat price of \$10 for up to 5 miles. Taxi B charges \$6 then \$1 per extra mile. Taxi C charges \$2 then \$3 per extra mile.

What is the best price that you can get while going the farthest possible, in other words, what is the lowest dollars-per-mile ratio possible?

This problem involves several lines. Let x=distance (miles) and y=price (\$). Then taxi A represents the line $y = 10$, taxi B represents $y = x + 6$ and taxi C is $y = 3x + 2$. Also there are several inequalities (like there are in most any optimization problem): $3 \leq x \leq 6$, and $y \leq 12$.

This can be made into a tropical polynomial $y = p(x) = 2 \odot x^3 \oplus 6 \odot x \oplus 10$. Then $\mathcal{H}(p) = \{2, 3, 5\}$, or in (x,y) coordinates, $\{(2, 8), (3, 9), (5, 10)\}$. The point $(2, 8)$ is not even a possibility because it is not true that $x \geq 3$. Out of the points (3,9) and (5,10), the latter is the better deal, because you are paying only \$2 per mile instead of \$3. This also meets all the requirements because $3 \leq 5 \leq 6$ and $10 \leq 12$. Of course (6,10) is a point on the graph and this is an even better deal, but remember that taxi A only goes for up to 5 miles. Therefore the answer is \$2 per mile is the best deal.

5 Bibliography

https://math.berkeley.edu/˜bernd/mathmag.pdf