DEDEKIND DOMAINS

JANE AHN

1. Background

Definition 1.1. *An **integral domain** is a commutative ring with an identity and no zero divisors.*

Definition 1.2. *An **integrally closed domain** is an integral domain whose fraction field is itself.*

Definition 1.3. *A **Noetherian ring** is a ring such that every ideal is finitely generated, its ideals satisfy the ascending chain condition (if $I_1 \subseteq I_2 \subseteq \dots$ is an ascending chain of ideals of R, then there exists a positive integer N such that for all $m, n > N$, $I_m = I_n$), and given a nonempty set S of ideals of R, S has a maximal element (i.e. there exists $I \in S$ such that for all $J \in S$, $I \not\subset J$).*

2. Dedekind Domain

Definition 2.1. *An integral domain R is a **Dedekind Domain** if it is Noetherian of dimension 1, and for all maximal ideals P of R, the localization R_p is a regular local ring.*

An alternate, and simpler, definition is the following:

Definition 2.2. *A domain B is a **Dedekind Domain** iff:*

(1) *B is Noetherian*
(2) *B is integrally closed*
(3) *$dim(B) = 1$ (which means nonzero prime ideals are maximal)*

An example of a Dedekind domain is $\mathbb{Z}$, the set of integers. Any principal ideal domain is a Dedekind domain because it is integrally closed and the nonzero prime ideals are maximal.

Lemma 2.0.1. ***Prime Avoidance Lemma*** *Let $P_1, P_2, \dots, P_s$ with $s \geq 2$ be ideals of a ring R with P_1 and P_2 not necessarily prime, but $P_3, \dots, P_s$ prime. Now let I be any ideal of R. The idea is that if you can avoid each P_j individually, i.e. for each j, find an element in I but not in P_j, then you can avoid all of the P_j simultaneously, i.e. find a single element in I that it is in none of the P_j.*

Proof. (Sketch) We will prove the contrapositive: if $I \subseteq \cup_{i=1}^{s} P_i$, then for some i, $I \subseteq P_i$.

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(1) Suppose the result is false. Without loss of generality, assume there exists $a_i \in I$ without $a_i \in P_i$, but $a_i \notin P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_s$.
(2) Prove for $s = 2$.
(3) Assume $s > 2$, and observe $a_1 a_2 \ldots a_{s-1} \in P_1 \cap \cdots \cap P_{s-1}$, but $a_s \notin P_1 \cup \cdots \cup P_{s-1}$. Let $a = (a_1 a_2 \ldots a_{s-1}) + a_s \notin P_1 \cup \cdots \cup P_{s-1}$. Show that $a \in I$, $a \notin P_1 \cup \cdots \cup P_s$, a contradiction.

□

Theorem 2.1. *Let B be a Dedekind domain with finitely many maximal ideals. Then B is a PID.*

Proof. Let $M_1, \ldots, M_g$ be maximal ideals of B. By the prime avoidance lemma, M_1 is not contained in $M_1^2 \cup M_2 \cup \cdots \cup M_g$ and there exists $t \in M_1$ such that $t \notin M_1^2$ and $t \notin M_i$ for $i \geq 2$. Now consider the ideal tB. This factors as a product of maximal ideals. Thus, there are integers $e_i \geq 0$ such that $tB = M_1^{e_1} \ldots M_g^{e_g}$. For $i \geq 2$, if $e_1 \geq 1$, then $tB \subseteq M_i^{e_i} \subseteq M_i$, which says $t \in M_i$. Since this is false, $e_i = 0$ for $i \geq 2$. Hence, $tB = M_1^{e_1}$. Since $t \notin M_1^2$, $e_1 < 2$. Hence, $e_1 = 1$ and $tB = M_1$. Therefore, M_1 is principal. Similarly, each of the M_i is principal. Finally, since every nonzero ideal of B is a product of maximal ideals, each ideal is principal. □

Now, recall that the product of ideals contain all finite sums of elements in the ideals.

Proposition 2.1.1. *If I is a nonzero ideal of the Noetherian integral domain R, then I contains a product of nonzero prime ideals.*

Proof. Suppose not. If S is the set of all nonzero ideals that do not contain a product of nonzero prime ideals, then as R is Noetherian, S has a maximal element J, and J cannot be prime because it is in S. Hence, there exists $a, b \in R$ such that $a, b \notin J$, but $ab \in J$. By maximality of J, ideals $J + Ra, J + Rb$ each contain product of nonzero prime ideals, so hence so does their product $(J + Ra)(J + Rb) \subseteq J + Rab = J$. Contradiction. □

Corollary 2.1.1. *If I is an ideal of the Noetherian ring R (not necessarily an integral domain), then I contains a product of prime ideals.*

Proof. Repeat the above proof with "nonzero" omitted. □