GRASSMANNIANS

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1. Introduction to Grassmannians

As we know, projective space $\mathbb{P}^n$ is defined as the set of lines through the origin in $n+1$ dimensions. This definition, however, can be further generalized. Instead of lines through the origin, we can use the set of k-dimensional planes through the origin. These sets are called Grassmannians.

Definition 1.1. A Grassmannian $G(k, n)$ is the set of k-dimensional planes through the origin in n-dimensional space.

Thus, the projective space $\mathbb{P}^n$ is also $G(1, n+1)$.

2. Wedge Products and the Exterior Power

It would be helpful to visualize Grassmannian space in terms of projective space. To do this, we must first define the *wedge product*.

Definition 2.1. The wedge product of two vectors is defined as an operation with the following properties:

1. $a \wedge a = 0$
2. $a \wedge b = -b \wedge a$.
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
4. $a \wedge (b + c) = a \wedge b + a \wedge c$
5. $ka \wedge b = k(a \wedge b) = a \wedge kb$

Definition 2.2. The wedge product of two vectors $a \wedge b$ is a *simple bivector*, or, alternatively, a *2-blade*.

A simple bivector is composed of a direction (a two-dimensional subspace) and a magnitude. Similar to vectors, simple bivectors can be multiplied by constants, scaling the magnitude. They can also be added together, satisfying Properties 1, 2, and 4 of the wedge product in Definition 2.1.

The closure of the simple bivectors in $\mathbb{R}^n$, under both scalar multiplication and addition, is the exterior power $\Lambda^2(\mathbb{R}^n)$. Not all elements of this closure are simple bivectors.

Example. In $\Lambda^2(\mathbb{R}^4)$, given basis u_1, u_2, u_3, u_4, the element $(u_1 \wedge u_2) + (u_3 \wedge u_4)$ cannot be expressed as a wedge product, and thus is a bivector that is not simple.

Proposition 2.3. *$\Lambda^2(\mathbb{R}^4)$ is a vector space with dimension 6.*

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Proof. By definition, all bivectors can be written as the linear combination of simple bivectors. All simple bivectors in $\Lambda^2(\mathbb{R}^4)$ can be written as $(c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4) \wedge (c_5u_1 + c_6u_2 + c_7u_3 + c_8u_4)$ where each c_i is a scalar. Using Property 4, the distributive property, and property 2, the anticommutative property, we get a linear combination of $(u_1 \wedge u_1)$, $(u_2 \wedge u_2)$, $(u_3 \wedge u_3)$, $(u_4 \wedge u_4)$, $(u_1 \wedge u_2)$, $(u_2 \wedge u_3)$, $(u_3 \wedge u_4)$, $(u_1 \wedge u_4)$, $(u_1 \wedge u_3)$, and $(u_2 \wedge u_4)$. By property 1, the first four terms go to zero, leaving the other six. Because all elements of $\Lambda^2(\mathbb{R}^4)$ can be written as linear combinations of these six elements, and all the elements are linearly independent, these six form a basis. Thus, $\Lambda^2(\mathbb{R}^4)$ has a basis of size 6, and also dimension 6.

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We can extend the definition of a simple bivector to more dimensions.

Definition 2.4. The wedge product of k vectors is a k-blade (simple k-vector). Similar to simple bivectors, they have a direction (the k-dimensional subspace containing them), and a scalar magnitude.

Like simple bivectors, the set of k-blades also has an exterior power.

Definition 2.5. The closure of k-blades in $\mathbb{R}^n$ is the exterior power $\Lambda^k(\mathbb{R}^n)$.

This space, like $\Lambda^2(\mathbb{R}^n)$, has elements that are not k-blades, however, all elements are linear combinations of k-blades, leading to a result similar to Proposition 2.3.

Proposition 2.6. $\Lambda^k(\mathbb{R}^n)$ *is a vector space with dimension* $\binom{n}{k}$*.*

The proof of this is quite similar to Proposition 2.3, with the set of wedge products of each k-sized subset of the basis vectors of $\mathbb{R}^n$ forming a basis for $\Lambda^k(\mathbb{R}^n)$. There are $\binom{n}{k}$ such subsets, and thus the space has dimension $\binom{n}{k}$. This means that $\Lambda^k(\mathbb{R}^n)$ is isomorphic to $\mathbb{R}^{\binom{n}{k}}$

3. The Plücker embedding

Given the fact that the exterior power $\Lambda^k(\mathbb{R}^n)$ contains elements consisting of a k-plane through the origin (k-dimensional subspace) and a scalar, it is easy to see a possible connection to Grassmannians. To understand the relationship clearly, consider the projective space, $\mathbb{P}(\Lambda^k(\mathbb{R}^n))$. As points in projective space are invariant to scaling, the scale factor in the k-blade disappears, leaving only the k-plane. This creates a direct map from the Grassmannian to $\mathbb{P}(\Lambda^k(\mathbb{R}^n))$. The image of this map are the projections of the k-blades, however, recall that not all k-vectors in $\Lambda^k(\mathbb{R}^n)$ are k-blades. Thus, the full map $G(k,n) \to \mathbb{P}(\Lambda^k(\mathbb{R}^n))$ is only an injective map, not a bijective one. This map is the Plücker embedding. The coefficients of the mapping of a Grassmannian point P with respect to the basis of $\Lambda^k(\mathbb{R}^n)$ are the *Plücker coordinates* of P.

Given that $\Lambda^k(\mathbb{R}^n)$ is isomorphic to $\mathbb{R}^{\binom{n}{k}}$ by Proposition 2.6, $\mathbb{P}(\Lambda^k(\mathbb{R}^n))$ is isomorphic to $\mathbb{P}^{\binom{n}{k}-1}$. Because of this, elements of $\mathbb{P}(\Lambda^k(\mathbb{R}^n))$ have coordinates in $\mathbb{P}^{\binom{n}{k}-1}$. The Plücker embedding then becomes a map $G(k,n) \to \mathbb{P}^{\binom{n}{k}-1}$, and the Plücker coordinates are simply the coefficients of the resulting vector.

Remark 3.1. Because the Plücker coordinates exist in projective space, they are only determined up to a scale factor.

3.1. **Calculation of the Plücker Embedding.**

To use the Plücker embedding on a point G in Grassmannian space, take a basis of the k-dimensional plane. This basis can be represented as a $n \times k$ matrix, where the matrix's column space is G. The $k \times k$ minors of this matrix, give us coordinates in $\mathbb{R}^{\binom{n}{k}}$. Because each element of the matrix can be multiplied by a constant while still spanning the same point in Grasmannian space, these coordinates are points in $\mathbb{P}^{\binom{n}{k}-1}$.

The Plücker coordinates do also satisfy certain quadratic relations, known as the Plücker relations. Points in $\mathbb{P}(\Lambda^k(\mathbb{R}^n))$ that satisfy the relations make up the image of Plücker embedding - namely, the resulting k-vector must be a k-blade. These relations, in fact, provide an alternate definition for a k-blade. For all points in $\mathbb{P}^{\binom{n}{k}-1}$ satisfying these relations, there is a bijective map from $G(k, n)$.

4. Flags and Schubert Cells

A major component of the study of Grassmannians is through subdivisions known as Schubert cells. In order to define Schubert cells, however, we first need to define flags.

Definition 4.1. A flag is a set of nested linear subspaces $A_1 \subset A_2 \subset ... \subset A_n$ of a vector space V such that $dim(A_n) < dim(A_{n+1})$ for all n.

A Schubert cell can be defined as a subset of a Grassmannian whose elements intersect a flag in a certain way.

Definition 4.2. Given a certain flag$A : A_1 \subset A_2 \subset ... \subset A_k$, where $dim(A_k) \leq n$ and a sequence of integers $0leqa_1 \leq a_2 ... \leq a_k$, its corresponding Schubert cell in $G(k, n)$ is the subset consisting of all points L such that $dim(L \cap A_i) \geq a_i$ for $i = 1, 2, ..., k$.

If instead $dim(L \cap A_i) = a_i$, those points form an open Schubert cell. Open Schubert cells are a generalized form of the affine patches in projective space - each of these open cells is isomorphic an affine space. Also, similar to the affine patches, for a given flag, the Grassmannian can be represented as the disjoint union of all Schubert cells in the flag.

Schubert cells are important to enumerative geometry, which is an extension of the study of intersections.Their study, known as Schubert calculus, is a part of the solution to Hilbert's fifteenth problem - to provide a rigorous formalization of enumerative geometry. The problem is currently partially solved, with Schubert calculus as an integral part of this solution,

5. The Amplituhedron

The concept of a convex polygon's interior can be defined in the projective plane P_2 as the projection of all possible linear combinations of the vertices. This definition can be generalized to all projective space, as well as to the Grassmannian, using each of the 'vertices' of the polygon in $\mathbb{P}(\Lambda^k(\mathbb{R}^n))$. The Tree Amplituhedron, given a set of linearly independent k-blades, consists of all k-blades which are positive linear combinations of the elements of the set. The Tree Amplituhedron has uses in modern advanced physics.

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