27 Lines on a Cubic Surface Presentation Notes

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1 Introduction

The objective of the presentation is to provide a sketch of the proof that there are 27 lines through a smooth cubic surface. A cubic surface S in is the vanishing set of a cubic polynomial $\mathbb{P}^3$. The outline of the proof will use several lemmas. Since all of them cannot be presented due to time constraints, some of the simpler or non essential ones will just be listed. However, note that there is one important lemma that is essential, but is too hard to prove today. This lemma states that there is AT LEAST one line through a smooth cubic surface. Let K be any algebraically closed field. Therefore, the outline given will only try to show that there are 27 lines on every smooth cubic surface with at least one line. Our cubic surface will be $S = V(f)$ in P^3 where $f \in K[x, y, z, t]$ is an irreducible homogeneous polynomial.

2 Lemmas on quadric surfaces

3 Lemmas on cubics

Lemma 3.1. *Let $P \subset P^3$ be a plane. Then the intersection of the surface S with the plane P forms one of the following in the plane P: (1) a nondegenerate cubic curve (namely, a cubic curve defined by an irreducible cubic form), (2) the union of a nondegenerate conic curve and a line, or (3) three distinct lines.*

Proof. In order to simplify our work, we'll first rotate the coordinates so that the plane P is defined as $V(t)$. Now we know that every point on the intersection has t coordinate 0. The intersection would be defined with the same cubic as f, except all terms with t would no longer exist. Let the new polynomial be $g \in K[x, y, z]$. Since g is a cubic, it could factor into one of the following: (i) it's irreducible, (ii) it factors into a line and a conic, (iii) it factors into three DISTINCT lines. Now, the proof that if it factors into lines, then the lines must be distinct is quite long, so we will leave it out of this outline. □

Lemma 3.2. *Let p be a point of S. Then the number of lines which are contained in S and contain p is at most 3. Additionally, the tangent space to S at p is a plane containing all such lines.*

Lemma 3.3. *3. Let l be a line contained in S. Then there are exactly 10 lines on S which intersect l. We can write these 10 lines as $l_1, \ldots, l_5, l'_1, \ldots, l'_5$ in such a way that for each $i \in 1, ..., 5$, (1) $l, l_i, and l'_i$ all lie in some plane Pi and (2) if $j \in 1, ..., 5$ $i then (l_i \cup l'_i) \cap (l_i \cup l'_i) = \phi$.*

Proof. If any line $m \neq l$ intersects l, then it must define a plane P_m containing l, m. If m, l lie in S, the plane defined by them must intersect at a third line because 2 lines are not possible by Lemma 3.1. Let this third line be m'. All three of these lines lie on the same plane, and since any two lines on a projective plane must intersect, all of these lines intersect pairwise. Now as long as we have 5 distinct planes that intersect the cubic, we'll have the 5 pairs of lines describes above. It turns out that there are exactly 5 such planes. However, this proof is also extremely long, so we'll skip this one too. □

So we have 11 of the 27 lines accounted for now. Consider a pair of lines m and n on S that don't intersect. This is clearly possible since we could just take m and n to be the lines from 2 different planes from the previous lemma. Since m is a line in S, there are pairs of lines on 5 different planes that intersect m. The same is true for n. However, not all these 10 lines that intersect m and n are distinct. Turns out, 5 of them are in common between the two. So we have m, the 10 lines m intersects, n, and the 5 additional lines n intersects. That gives a current total of 17 lines. A longer lemma involving Lemma 3.2 shows that given any 5 lines that intersect m that don't intersect each other (l_{15} for example) or n we can find a new line that we haven't already accounted for that intersects any 3 of those. So we have another 5C3 lines. Adding all this up we get : $1 + 10 + 1 + 5 + 10 = 27$ lines through this cubic. That serves as a brief sketch of the proof, but there remains a lot left to be proven. We haven't proven many of the lemmas that we have used because they are quite long. This proof outline was taken from a proof by Simon Lazarus, which can be found here: http://math.uchicago.edu/ may/REU2014/REUPapers/Lazarus.pdf.