Tropical Algebra and Geometry

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1 Introduction

Tropical Algebra and Geometry are relatively newly developed fields of mathematics. Introduced by Hungarian-born Brazilian mathematician Imre Simon, the French mathematicians decided to name the field tropically, their image of Brazil simply constituting with the word tropical. In this paper, we will introduce the basics of tropical algebra and geometry.

2 Basics: Tropical Arithmetic

2.1 Operations

Let us start by defining the tropical world. The tropical semifield in defined to be $(\mathbb{R} \cup -\infty, \oplus, \odot)$. We are working with the real numbers and -∞ with the tropical addition and multiplication operations. We redefine addition and multiplication as follows:

$$x \oplus y := \max(x, y)$$

$$x \odot y := x + y$$

So the tropical sum is the maximum and the tropical product is the sum. For example,

$$2 \oplus 9 = 9$$

$$2 \odot 9 = 11$$

Logically, after defining tropical addition and multiplication we will look towards a tropical subtraction and division. However, tropical subtraction would be ill defined. For example,say we want to find $5 \ominus 6$. Let us start by assuming it equals x. Then, $5 \ominus 6 = x$ and therefore $x \oplus 6 = 5$. Since we have defined tropical addition as the maximum, there exists no x for which this equation is true and therefore no defined quotient. Division, however, is well-defined.

2.2 Properties

Both tropical addition and multiplication is commutative. I.E.

$$x \oplus y = y \oplus x$$

$$x \odot y = y \odot x$$

In addition, tropical multiplication is distributive over tropical addition. I.E.

$$a \odot (b \oplus c) = a \odot b \oplus a \odot c$$

Tropical arithmetic follows the usual order of operations and therefore tropical multiplication comes before tropical addition. Both operations also have identity elements:

$$x \oplus -\infty = x$$

$$x \odot 0 = x$$

2.3 Tropical Polynomials

Let us start by letting $x_1, \dots, x_n$ be variables in the tropical semifield. A *tropical monomial* is any product of these variables, where repetition is allowed, much like what we are used to.We can use the common notation of variables written to exponents, just remember that we are referring to tropical multiplication. So, for example,

$$x_1 \odot x_3 \odot x_3 \odot x_2 \odot x_4 \odot x_1 \odot x_1 \odot x_2 = x_1^3 x_2^2 x_3^2 x_4.$$

Note that this is not restricted to positive exponents (remember tropical division is defined). Also note that this tropical monomial is a linear function with integer coefficients in classical arithmetic as tropical multiplication is classical addition. Naturally it follows that a *tropical polynomial* is a combination of tropical monomials:

$$p(x_1, \dots, x_n) = a \odot x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \oplus b \odot x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} \oplus \dots$$

where coefficients $a, b, \dots$ are real numbers and $i_1, j_1, \dots$ are integers. A tropical polynomial is a function $\mathbb{R}^n \to \mathbb{R}$. If we convert the tropical polynomial to include only classical operations, we get the maximum of a finite collection of linear functions.

$$p(x_1, \dots, x_n) = \max(a + i_1 x_1 + \dots + i_n x_n, b + j_1 x_1 + \dots + j_n x_n, \dots).$$

Proposition 1. *The tropical polynomials in n variables $x_1, \dots, x_n$ are precisely the piecewise-linear concave functions on $\mathbb{R}^n$ with integer coefficients.*

The important thing to note about tropical polynomials is that each tropical polynomial *function* can be written uniquely as a product of tropical linear functions. So the Fundamental Theorem of Algebra holds in the tropical world. However, this is referring to functions. Multiple polynomials can be represented the same function. Therefore, each polynomial individually will not have a unique factorization but rather each polynomial can be written as an equivalent polynomial, representing the same function, that can be factored into linear factors. For example, the following equivalent polynomials (representing the same function) have the same factorization:

$$x^2 \oplus 17 \odot x \oplus 2 = x^2 \oplus 1 \odot x \oplus 2 = (x \oplus 1)^2$$

Unique factorization of polynomials no longer holds in two or more variables.

2.4 Tropical Algebra's Binomial Theorem

The coefficients for the polynomials generated by any $(a \oplus b)^n$ will all be zero. This is because our tropical sum is the maximum and therefore having multiple of the same term cannot alter the tropical sum. This neat fact for binomial expansion leads to a favorable result- the freshman's dream which states

$$(x \oplus y)^n = x^n \oplus y^n$$

To see this, we can look at the expansion of $(x \oplus y)^3$.

$$(x \oplus y)^3 = (x \oplus y) \odot (x \oplus y) \odot (x \oplus y)$$

$$= 0 \odot x^3 \oplus 0 \odot x^2 y \oplus 0 \odot xy^2 \oplus 0 \odot y^3$$

Since tropical multiplication is just normal addition, we can drop the zeros.

$$(x \oplus y)^3 = x^3 \oplus x^2 y \oplus xy^2 \oplus y^3$$

Now notice that the maximum of $x^3 \oplus x^2 y \oplus xy^2 \oplus y^3$ has to be either x^3 or y^3. This is because, if either x or y is greater, x^3 or y^3 will definitely be greater than $x^2 y$ and xy^2 which include the smaller term. And, if x and y are equal all four terms will be equal. Therefore, we can say that,

$$(x \oplus y)^3 = x^3 \oplus y^3$$

and more generally that

$$(x \oplus y)^n = x^n \oplus y^n$$

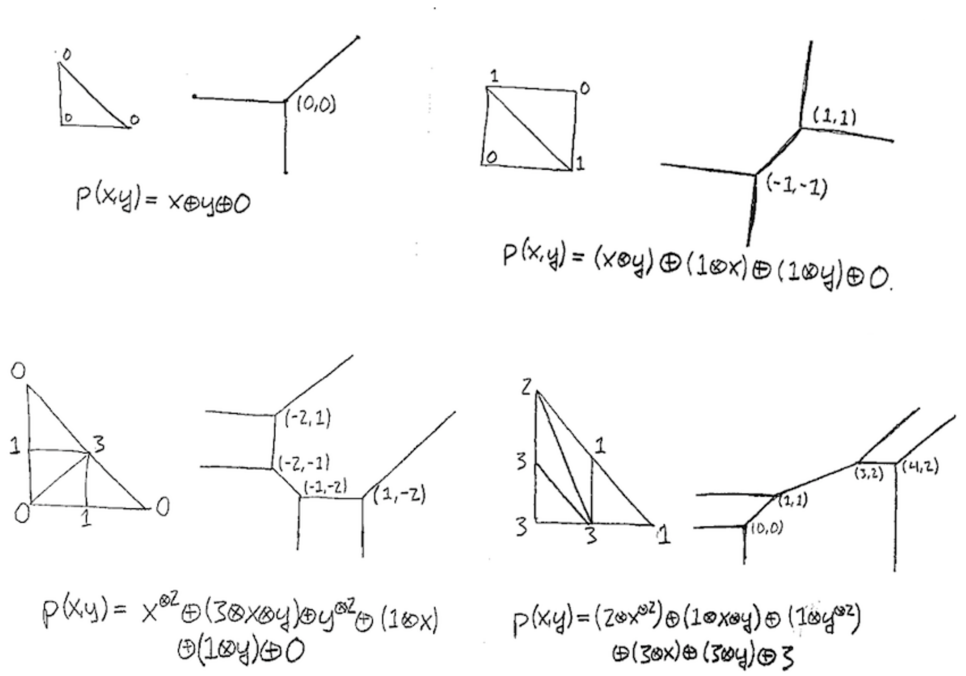

Figure 1: Four tropical curves with their Newton subdivisions to the left

3 Tropical Curves

3.1 Introduction

Consider a polynomial in two variables

$$p(x,y) = \bigoplus_{j,k} c_{jk} \odot x^j \odot y^k = \max_{j,k}(c_{jk} + jx + ky).$$

Definition 3.1. *Let $p(x,y)$ be as above. Then for all $(x,y) \in \mathbb{R}^2$, define the valence of (x,y) to be the number of monomials which achieve the value $p(x,y)$.*

Definition 3.2. *The tropical plane curve corresponding to $p(x,y)$ is the closure in $\mathbb{R}^2$ of the set of points with valence greater than 1. A subspace of a tropical plane curve is called an edge if it is a maximal subspace homeomorphic to an open interval, and points that do not lie on edges are called vertices.*

In other words, the tropical curve C defined by p consists of the points $(x,y) \in \mathbb{R}$ where p is not differentiable because maximum is assumed by more than one of the terms of p. Tropical polynomials are piecewise-linear functions with integer slope where differentiable.

3.2 Newton Subdivision

Definition 3.3. *Let $p(x,y)$ be a tropical polynomial, and suppose that the indices (i,j) of the monomial set are plotted in $\mathbb{Z}^2$. Then, the convex hull of these points is called the Newton polygon of p. Suppose in addition that an edge is drawn between all pairs of indices such that the corresponding monomials simultaneously achieve the value $p(x,y)$ somewhere in the plane. The result is a subdivision of the Newton polygon into small polygons. This subdivision is called the Newton subdivision, and the smaller polygons are called the faces of the subdivision.*

Newton subdivision appears to be the most efficient way to quickly visualize and draw tropical plane curves. Figure 1 shows the tropical curves in $\mathbb{R}^2$ corresponding to four different tropical polynomials. The Newton subdivisions are drawn to the left of each curve.

All of the examples of tropical plane curves that we have presented so far in figure 1 have had had fully subdivided Newton polygons. This is where the subdivision breaks the polygon into triangles of area $\frac{1}{2}$.

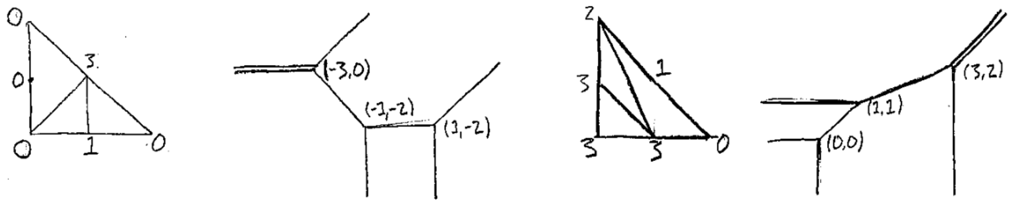

Figure 2: Two tropical curves whose Newton subdivision (to the left) do not fully subdivide the Newton polygon

However, consider what happens when this is not the case. For example, we can deform the bottom two tropical curves in figure 1 by changing a single coefficient to bring two of the edges into coincidence.The result is shown in figure 2. Clearly we should regard the edges that are drawn double in this figure as being edges of multiplicity 2.

3.3 Multiplicity

Before we define multiplicity, we must introduce $val(x)$.

Definition 3.4. *The field of Puiseux series, which we denote* $\mathbf{K}$*, is given by*

$$\bigcup_{k\geq 1} \mathbf{C}((t^{\frac{1}{k}})),$$

where the union is taken in an algebraic closure of $\mathbf{C}((t))$*. For any* $x \in \mathbf{K}^*$*, we denote by* $val(x)$ *the smallest exponent of* t *occurring in* x*.*

Definition 3.5. *The multiplicity of an edge in a hypersurface is the* $val(p) - 1$*, where* p *is any point on the interior of the edge.*

Edge multiplicity is very easy to read off of the Newton polygon: each edge in the hypersurface passes through some number of vertices of the polygon, and the multiplicity is one less than this number.

In order to specify a tropical plane curve, we should really give not only the set in $\mathbb{R}^2$, but also the edge multiplicities. These data uniquely determine the polynomial defining the curve, up to tropical multiplication by a scalar.