EUCLIDEAN DISTANCES

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Abstract. In this paper, we cover the constraints regarding the pairwise distances of points in the Euclidean space. We also cover a portion of linear algebra regarding eigenvalues and eigenvectors and matrix factorization.

1. The Triangle Inequality

It is a well known fact that the pairwise distances between three points must satisfy the triangle inequality.

Theorem 1.1. *For any three points* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *in any Euclidean space,*

$$||\mathbf{a} - \mathbf{b}|| \leq ||\mathbf{b} - \mathbf{c}|| + ||\mathbf{c} - \mathbf{a}||$$

where $||\mathbf{R}||$ *is the distance from* $\mathbf{R}$ *to the origin.*

It is also well known that the triangle inequality is the only restriction of the distances of three points. Therefore, for any three numbers, x, y, z, that satisfy $x \leq y + z$, $y \leq z + x$, and $z \leq x + y$, there exist points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^n$ where $n \geq 2$ such that $||\mathbf{a} - \mathbf{b}|| = x$, $||\mathbf{b} - \mathbf{c}|| = y$, and $||\mathbf{c} - \mathbf{b}|| = z$. We know this works for three points, but what happens when we have more than three points? In Figure 1, it can be observed that the distances between each group of three points comply with the triangle inequality, but there does not exist points $a, b, c, d \in \mathbb{R}^3$ such that this can happen.

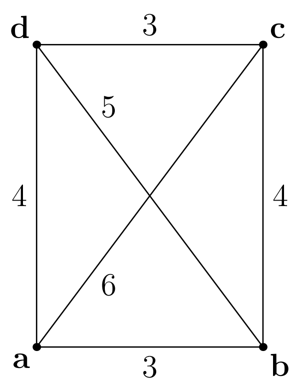

Figure 1. An example of an impossible figure.

One may try to prove that there is no common intersection between the three spheres around three of the points with radii corresponding to their distances with the other fourth point, but linear algebra provides a more elegant method of determining if the pairwise distances given is achievable.

2. Eigenvectors and Eigenvalues

We introduce the eigenvector and the eigenvalue.

Definition 2.1. An *eigenvector* of a transformation matrix A is a vector x such that $Ax = \lambda x, \lambda \in \mathbb{R}$, or, after multiplied by matrix M, is a scalar multiple of itself.

Definition 2.2. The eigenvectors are determined by solving for their corresponding λ first, which is called the *eigenvalue*.

We solve for the eigenvalues by using the **characteristic equation** of the matrix A which states

$$det(A - \lambda I) = 0.$$

Computing $A - \lambda I$ is the same as subtracting λ from each element on the diagonal of A. Therefore, if $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$, then $A - \lambda I = \begin{bmatrix} a_{11} - \lambda & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} - \lambda \end{bmatrix}$. We now simply solve for λ from $det(A - \lambda I) = 0$. Since there are n rows, the degree of the polynomial created on the LHS will be n, thus, giving n solutions. To solve for the eigenvectors, we must first rearrange this equation:

$$\begin{aligned} Ax &= \lambda x \\ Ax - \lambda x &= 0 \\ (A - \lambda I)x &= 0 \end{aligned}$$

Plug in for each λ and $A - \lambda I$ becomes a matrix with constant elements. We now simply let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and expand the multiplication, creating a system of linear equations.

Remark 2.3. For a unique λ, the solutions for eigenvectors has one degree of freedom since multiplying the vector by scalar is the equivalent of muliplying the scalar after multiplying the matrix, but for each duplicate, the degree of freedom increases.

3. Diagonalization of Symmetric Matrices

We first introduce the concept of the basis, then we use the basis of eigenvectors to diagonalize a matrix.

Definition 3.1. The *basis* is a coordinate system that relies on a set of n vectors each of size n. A vector $\vec{v}$, if $\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_n\vec{v_n}$, then $\vec{v}$ in basis $\mathcal{A} = (\vec{v_1}, \vec{v_2}, \cdots \vec{v_n})$, represented as $[\vec{v}]_{\mathcal{A}}$ is equal to $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. For example, $\mathbb{R}^3$ uses the basis $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$. The vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in basis $\mathcal{U} = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)$ is equal to $\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$ since

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Theorem 3.2. *For vector $\vec{x}$ and basis $\mathcal{B} = (\vec{v_1}, \vec{v_2}, \cdots \vec{v_n})$, $\vec{x} = S[\vec{x}]_{\mathcal{B}}$ where $S = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$.*

Proof. Let $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. Then,

$$\vec{x} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_n\vec{v_n} = [\vec{v_1}, \vec{v_2}, \cdots \vec{v_n}] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = S[\vec{x}]_{\mathcal{B}}.$$

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Theorem 3.3. *Let there be a transformation matrix A. Let the matrix B be the $\mathcal{B}$-matrix of A, which performs the transformation of A in basis $\mathcal{B} = (\vec{v_1}, \vec{v_2}, \cdots \vec{v_n})$ (i.e. $[A\vec{x}]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$). Then, $AS = SB$ where $S = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$.*

Proof. We know that $AS[\vec{x}]_{\mathcal{B}} = A\vec{x}$. We also know that $AS[\vec{x}]_{\mathcal{B}} = S[A\vec{x}]_{\mathcal{B}} = A\vec{x}$. Therefore,

$$AS[\vec{x}]_{\mathcal{B}} = AS[\vec{x}]_{\mathcal{B}} \Longrightarrow AS = SB.$$

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We now move on to the diagonalization of a matrix.

Theorem 3.4. *Let there be a matrix A and eigenbasis $\mathcal{D} = (\vec{v_1}, \vec{v_2}, \cdots \vec{v_n})$ of A with $A\vec{v_i} = \lambda_i\vec{v_i}$. Then the $\mathcal{D}$-matrix D of A is*

$$D = S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}, \textit{ where } S = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$$

Proof. We know that $D = S^{-1}AS$ is true from rearranging Theorem 3.3. We must prove that the $\mathcal{D}$-matrix of A is the diagonal matrix with the eigenvalues of A. Let there be a vector $\vec{x} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_n\vec{v_n}$. Then, $D[x]_{\mathcal{D}} = [Ax]_{\mathcal{D}}$. We see that

$$\begin{aligned} Ax &= A(c_1\vec{v_1} + c_2\vec{v_2} + \cdots c_n\vec{v_n}) \\ &= c_1A\vec{v_1} + c_2A\vec{v_2} + \cdots c_nA\vec{v_n} \\ &= c_1\lambda_1\vec{v_1} + c_2\lambda_2\vec{v_2} + \cdots c_n\lambda_n\vec{v_n}. \end{aligned}$$

Therefore, $[Ax]_{\mathcal{D}} = \begin{bmatrix} c_1\lambda_1 \\ c_2\lambda_2 \\ \vdots \\ c_n\lambda_n \end{bmatrix}$. We know that $D[x]_{\mathcal{D}} = D \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. Thus, since $D[x]_{\mathcal{D}} = [Ax]_{\mathcal{D}}$ it is obvious that

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$

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4. Matrix Factorization

We present a few ways to factorize matrices.

Theorem 4.1. *A positive semidefinite real matrix has eigenvalues that are nonnegative.*

Proof. Let the matrix be A. Since it is positive semidefinite, $x^T MX \geq 0$. Let $\vec{v}$ be an eigenvector of A. Then,

$$v^T Mv = v^T \lambda v = v^T v\lambda \geq 0.$$

Since $v^T v$ must be nonnegative, λ, which is the eigenvalue of $\vec{v}$ must also be nonnegative. ■

This will come in use later.

Definition 4.2. A real matrix M is said to be *positive semidefinite* iff, $\forall x \in \mathbb{R}^n, x^T Mx \geq 0$, where M^T is the transpose of M.

Theorem 4.3. *A real symmetric $n \times n$ matrix A is positive semidefinite if there exists an $n \times n$ real matrix X such that $A = X^T X$.*

Proof. Since $A = X^T X$, we know that $x^T Ax = x^T X^T Xx = (Xx)^T(Xx) = ||Xx||^2 \geq 0$.
We now prove the other direction:
Let A be a real symmetric $n \times n$ matrix. Therefore, it is positive semidefinite and is diagonalizable: $D = S^{-1}AS \implies A = SDS^{-1}$ where $S = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix}$ and $\vec{v_1}, \vec{v_2}, \cdots \vec{v_n}$ are the eigenvectors of A and D is a diagonal matrix containing the eigenvalues of A. Let $T = S^{-1}$, so $A = T^{-1}DT$.

Theorem 4.4. *The eigenvectors of a symmetric matrix are orthogonal.*

Proof. For any matrix A and vectors $\mathbf{x}$ and $\mathbf{y}$, we know that

$$\langle A\mathbf{x}, \mathbf{y}\rangle = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = \langle \mathbf{x}, A^T\mathbf{y}\rangle.$$

Now let A be a symmetric matrix and $\mathbf{x}$ and $\mathbf{y}$ be its eigenvectors. Let the corresponding distinct eigenvalues be λ and μ, respectively. Therefore,

$$\lambda\langle \mathbf{x}, \mathbf{y}\rangle = \langle \lambda\mathbf{x}, \mathbf{y}\rangle = \langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^T\mathbf{y}\rangle = \langle \mathbf{x}, A\mathbf{y}\rangle = \langle \mathbf{x}, \mu\mathbf{y}\rangle = \mu\langle \mathbf{x}, \mathbf{y}\rangle.$$

Thus, $\lambda\langle \mathbf{x}, \mathbf{y}\rangle = \mu\langle \mathbf{x}, \mathbf{y}\rangle \implies (\lambda - \mu)\langle \mathbf{x}, \mathbf{y}\rangle = 0$. Since $\lambda \neq \mu$, then $\lambda - \mu \neq 0$ and $\langle \mathbf{x}, \mathbf{y}\rangle = 0$ implying that $\mathbf{x} \perp \mathbf{y}$. ■

Therefore, T is orthogonal and $T^{-1} = T^T$, so $A = T^T DT$.
Let $R = \sqrt{D} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{\lambda_n} \end{bmatrix}$. Since all eigenvalues are nonnegative by Theorem 4.1, R remains a real matrix. Define the matrix $X = RT$. We can now see that $X^T X = (RT)^T RT = T^T R^T RT = T^T DT = A$. ■

5. The Theorem for Pairwise Distances of Several Points

We finally present and prove the following theorem.

Theorem 5.1. *Let $m_{ij}, i, j = 0, 1, \ldots, n$, be nonnegative real numbers with $m_{ij} = m_{ji}$ for all i, j and $m_{ii} = 0$ for all i. Then points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{R}^n$ with $||\mathbf{p}_i - \mathbf{p}_j|| = m_{ij}$ for all i, j exist iff the $n \times n$ matrix G with*

$$g_{ij} = \frac{1}{2}\left(m_{oi}^2 + m_{0j}^2 - m_{ij}^2\right)$$

is positive semidefinite.

Proof. Let the given points be $\mathbf{p_0}, \mathbf{p_1}, \ldots \mathbf{p_n}$ in $\mathbb{R}^n$. The *cosine theorem* tells us that for any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $||\mathbf{x} - \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle$ where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the dot product of $\mathbf{x}$ and $\mathbf{y}$. We define $\mathbf{x_i} := \mathbf{p_i} - \mathbf{p_0}$ for $i = 1, 2, \ldots n$. By substituting and rearranging the cosine theorem, we see that $\langle \mathbf{x_i}, \mathbf{x_j} \rangle = \frac{1}{2}(||\mathbf{x_i}||^2 + ||\mathbf{x_j}||^2 - ||\mathbf{x_i} - \mathbf{x_j}||^2) = g_{ij}$. Therefore, G is the *Gram Matrix* of vectors $\mathbf{x_i}$, which is a matrix in which each entry is given by $g_{ij} = \langle \mathbf{v_i}, \mathbf{v_j} \rangle$ when it is of vectors $v_1, v_2, \ldots v_n$. Therefore, it is clear that G can be represented as $X^T X$. Thus, G is positive semidefinite.

We now prove the other direction of the "if and only if". If G is positive semidefinite, it can be represented as $G = X^T X$ for some $n \times n$ matrix X. Let X consist of column vectors $\mathbf{p_k}$ where $\mathbf{p_i} \in \mathbb{R}^2$ is in the ith column for $i = 1, 2, \cdots n$. Define $\mathbf{p_0} := 0$. Therefore,

$$g_{ii} = \langle x_i, x_i \rangle = \langle p_i - p_0, p_i - p_0 \rangle = ||p_i - p_0||^2 = m_{0i}^2$$

and

$$g_{ij} = \langle x_i, x_j \rangle = \frac{1}{2}\left(||x_i||^2 + ||x_j||^2 - ||x_i - x_j||^2\right) = \frac{1}{2}\left(m_{0i}^2 + m_{0j}^2 - ||p_i - p_j||^2\right)$$

This implies $||p_i - p_j|| = m_{ij}$ and the proof is complete. ∎

References

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