INFINITUDE OF PRIMES USING P-SERIES AND THE EULER PRODUCT

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1. Introduction

The prime numbers have been studied for thousands of years. More specifically, proving that infinitely many prime numbers exist has been a clear area of focus. Eratosthenes first came up with a way to "filter" out which numbers are prime among the first few integers using his Sieve of Eratosthenes. Euclid gave the first proof of the infinitude of primes 2000 years ago, as shown below:

Theorem 1.1. *There exist infinitely many prime numbers.*

Proof. Assume there are finitely many prime numbers. We list these prime numbers out to be p_1, p_2,…, p_k. for some finite number k. This implies that every other integer to exist is a composite number. These composite numbers must be divisible by at least one of our existing prime numbers. However, consider the number $N = p_1 * p_2 * \cdots * p_k + 1$. By definition, it must be composite, since it is not one of the original prime numbers. Notice the following:

$$N = p_1 * p_2 * \cdots * p_k + 1 \equiv 1 \bmod p_1$$
$$N = p_1 * p_2 * \cdots * p_k + 1 \equiv 1 \bmod p_2$$
$$\vdots$$
$$N = p_1 * p_2 * \cdots * p_k + 1 \equiv 1 \bmod p_k$$

N is not divisible by any of our prime numbers, therefore we conclude N must be prime. However, we declared it as composite and we have reached a contradiction. It follows that our original condition of a finite number of primes is wrong. There exist infinitely many prime numbers. ∎

Following the above proof, know that there exist infinitely many prime numbers. Nowadays, we look for more intriguing, clever and revealing proofs of the existence of infinitely many prime numbers. This paper will discuss how to use a certain type of series and a famous product in order to prove the infinitude of primes.

2. Series Divergence and Convergence

Definition 2.1. The *harmonic series* is defined as below:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

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Theorem 2.2. *The harmonic series is divergent.*

Proof. We can write out the harmonic series as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

We know the following is true:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots > \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \dots$$

We can combine "like terms" on the RHS of the inequality to yield:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \dots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

We know the RHS diverges, and since our harmonic series has a sum that is greater than this divergent series, we know the harmonic series diverges. ■

Definition 2.3. The p-series is defined as below:

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

for some positive integer s. Note that the harmonic series is the subcase of $s = 1$.

Theorem 2.4. *The p-series is convergent for $s > 1$.*

Proof. Like we did for the harmonic series, we should first write out a couple of terms for the main p-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

Notice that for all $s > 1$, the function $\frac{1}{n^s}$ is decreasing. We can use a neat result to help us out:

Lemma 2.5. *For some function f, decreasing on the interval of $[1, \infty)$, we have that $\sum_{n=1}^{\infty} f(n)$ converges if and only if its respective integral, $\int_1^{\infty} f(x)dx$ converges.*

This result is commonly known as the Integral Test. We can use the Integral Test for our p-series, because, as mentioned before, $\frac{1}{n^s}$ is a decreasing function for $s > 1$. We have that:

$$\int_1^{\infty} \frac{1}{x^s} dx = \frac{-1}{x^{s-1}(s-1)}\Big|_1^{\infty} = \lim_{x \to \infty} [\frac{-1}{x^{s-1}(s-1)}] + \frac{1}{s-1} = \frac{1}{s-1}$$

Therefore, since the respective integral of the p-series converges, we know that the p-series itself converges. ■

3. Euler Product

Definition 3.1. We define the Euler product as below:

$$\prod_{p}^{\infty} \frac{1}{1-p^{-s}}$$

where p starts at 2, and iterates through all of the prime numbers.

Definition 3.2. We define the Riemann Zeta Function as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for complex values of s. Note that the p-series is a subcase of the Riemann Zeta Function, where s is a real number.

Conjecture 3.3. *The Riemann Zeta Function and the Euler Product are equal for $s > 1$:*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p}^{\infty} \frac{1}{1-p^{-s}}$$

Proof. We write out the first few terms of the zeta function, then we perform the following algorithm:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \dots$$
$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} + \dots$$

Subtract the above two equations, then we perform the algorithm again:

$$(1-\frac{1}{2^s})\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \dots$$
$$\frac{1}{3^s}(1-\frac{1}{2^s})\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \dots$$

Again we subtract to yield:

$$(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \dots$$

Notice how our algorithm is behaving quite similar to the Sieve of Eratosthenes. Consider the end of each time we use the algorithm. For any term on the left that is expressed as $(1-\frac{1}{p^s})$ for some prime p, and for any term on the right that is expressed as $\frac{1}{q^s}$ for some integer q, we know that $\gcd(p,q) = 1$. In other words, p and q are relatively prime, implying that we are filtering out all of the prime numbers and their multiples each time we perform the algorithm for the next prime number. Therefore, we can expect that after performing the algorithm an infinite number of times for all the prime numbers, we see that:

$$\dots(1-\frac{1}{11^s})(1-\frac{1}{7^s})(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s) = 1$$

$$\zeta(s) = \frac{1}{\dots(1-\frac{1}{11^s})(1-\frac{1}{7^s})(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})} = \prod_p^\infty \frac{1}{1-p^{-s}}$$

■

4. Primes and the Riemann Zeta Function: Tying Them Together

Proposition 4.1. *There are infinitely many prime numbers.*

There are many different proofs as to why there are infinitely many prime numbers. We will be investigating the one that brings prime numbers, the Riemann zeta function, and the Euler product together.

Claim 4.2. *We can use the Riemann Zeta Function to further justify our infinitude of primes argument.*

Proof. We have the Riemann Zeta Function and Euler Product equality, as shown below:

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_p^\infty \frac{1}{1-p^{-s}}$$

Consider the $\zeta(1)$, which is the harmonic series:

$$\zeta(1) = \sum_{n=1}^\infty \frac{1}{n} = \prod_p^\infty \frac{1}{1-p^{-1}} = \prod_p^\infty \frac{p}{p-1}$$

Recall that the harmonic series is divergent, meaning that it cannot take on the value of a number. Let us expand out $\prod_p^\infty \frac{p}{p-1}$, since we have so far proved that this is equal to the harmonic series:

$$\sum_{n=1}^\infty \frac{1}{n} = \prod_p^\infty \frac{p}{p-1} = \frac{2}{1}\frac{3}{2}\frac{5}{4}\frac{7}{6}\frac{11}{10}\dots$$

Now, assume there are finitely many prime numbers. This would mean that the RHS of the above equation would be a convergent, finite number. However, the harmonic series is divergent. So, this would mean that a divergent series would converge to a value, a clear contradiction. Therefore, it must be that there are infinitely many prime numbers. ■

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