THE HADAMARD DETERMINANT PROBLEM

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1. Introduction

The Hadamard determinant problem is about how large the determinant of a matrix with entries equal to ± 1 can be. It was first introduced by Hadamard in his 1893 paper [Had93], where he proved a well-known bound on these determinants. Although some partial progress has been made, the problem of determining when this bound can actually be attained is still unsolved.

In this paper, we introduce and prove the spectral theorem, and explain how it can be used to prove Hadamard's bound. We also prove some other bounds on determinants of matrices with entries equal to ± 1. We will assume knowledge of basic linear algebra.

2. The Spectral Theorem and the Hadamard Determinant Problem

In this section, we state and prove the spectral theorem, and explain how it can be used to prove a bound on matrix determinants. Our motivating question is the following:

Question 2.1. *How large can* $\det(A)$ *be on the set of* $n \times n$ *matrices* $A = (a_{ij})$ *where* $|a_{ij}| \leq 1$ *for all* i, j*?*

While the determinant of a matrix can be fairly complicated to calculate, the determinant of a diagonal matrix is more simple, since it's just the product of the diagonal entries. We can use this to prove bounds on the determinant. We use the following well-known theorem:

Theorem 2.2. *For every real symmetric* $n \times n$ *matrix* A*, there exists a real orthogonal matrix* Q *such that* $Q^T AQ = Q^{-1}AQ$ *is diagonal.*

The standard proofs of this theorem use induction on n. We present an elegant proof from [AZ18], which uses a compactness argument. We first need the following lemmas for the proof of this theorem:

Lemma 2.3. *Let* $O(n)$ *be the set of real* $n \times n$ *orthogonal matrices. Then,* $O(n)$ *is a compact set when considered as a subset of* $\mathbb{R}^{n^2}$.

We will prove this lemma using the following well-known theorem:

Theorem 2.4 (Heine–Borel)**.** *Every closed and bounded subset of* $\mathbb{R}^N$ *is compact.*

Proof of Lemma 2.3. We want to show that $O(n)$ is closed and bounded. To show that it is closed, note that it is the preimage of the closed set $\{I\}$ under the continuous map $M \to MM^T$. To show that it is bounded, note that orthogonal matrices have orthogonal vectors as columns and rows, so for each $1 \leq i, j \leq n$, $|a_{ij}| \leq 1$. Thus, $O(n)$ is bounded, and by the Heine–Borel Theorem, it is compact. □

Our second lemma for the proof of Theorem 2.2 is the following:

Lemma 2.5. *For any real matrix A, let $\mathrm{Od}(A) = \sum_{i \neq j} a_{ij}^2$ be the sum of the squares of the off-diagonal entries. Note that if $\mathrm{Od}(A) = 0$, then A is diagonal.*

Now let $A = (a_{ij})$ be a real symmetric $n \times n$ matrix with $\mathrm{Od}(A) > 0$. Then, there exists some $U \in O(n)$ such that $\mathrm{Od}(U^T AU) < \mathrm{Od}(A)$.

Proof. Since $\mathrm{Od}(A) > 0$, there exists some r, s such that $a_{rs} \neq 0$. Then, let U be the identity matrix I_n, except $u_{rr} = u_{ss} = \cos\theta$, $u_{rs} = \sin\theta$, and $u_{sr} = -\sin\theta$:

$$\begin{bmatrix} 1 & 0 & & & \cdots & \cdots & & & 0 \\ 0 & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \cos\theta & & \sin\theta & & & \\ \vdots & & & & 1 & & & & \vdots \\ \vdots & & & & & \ddots & & & \vdots \\ & & & & & & 1 & & \\ & & & -\sin\theta & & \cos\theta & & & \\ & & & & & & & 1 & \\ & & & & & & & \ddots & 0 \\ 0 & & & & \cdots & \cdots & & 0 & 1 \end{bmatrix}$$

Notice that the columns of U are orthogonal unit vectors, so the matrix U is orthogonal. We compute the entries b_{ij} of $B = U^T AU$. We have

$$b_{kl} = \sum_{1 \leq i,j \leq n} u_{ik} a_{ij} u_{jl}$$

If neither k nor l is equal to r or s, then this is equal to a_{kl}, since $u_{ik} a_{ij} u_{jl}$ is only positive when $i = k$ and $j = l$.

Now we compute b_{kr} and b_{ks} if $k \neq r, s$: we have

$$b_{kr} = \sum_{1 \leq i,j \leq n} u_{ik} a_{ij} u_{jl} = \sum_{i=1}^{n} u_{ik}(a_{ir}\cos\theta - a_{is}\sin\theta) = a_{kr}\cos\theta - a_{ks}\sin\theta$$

and similarly,

$$b_{ks} = \sum_{1 \leq i,j \leq n} u_{ik} a_{ij} u_{jl} = \sum_{i=1}^{n} u_{ik}(a_{ir\sin\theta + a_{is}\cos\theta}) = a_{kr}\sin\theta + a_{ks}\cos\theta$$

This means that

$$b_{kr}^2 + b_{ks}^2 = (a_{kr}\cos\theta - a_{ks}\sin\theta)^2 + (a_{kr}\sin\theta + a_{ks}\cos\theta)^2$$
$$= a_{kr}^2\cos^2\theta - 2a_{kr}a_{ks}\sin\theta\cos\theta + a_{ks}^2\sin^2\theta + a_{kr}^2\sin^2\theta + 2a_{kr}a_{ks}\sin\theta\cos\theta + a_{ks}^2\cos^2\theta$$
$$= a_{kr}^2 + a_{ks}^2.$$

We can show in the same way that

$$b_{rl}^2 + b_{sl}^2 = a_{rl}^2 + a_{sl}^2$$

for $l \neq r, s$. This means that for any value of θ, $\mathrm{Od}(A)$ and $\mathrm{Od}(U^T AU)$ are equal except for the terms at (r, s) and (s, r).

We now show that we can choose some θ such that

$$\mathrm{Od}(U^T AU) = \mathrm{Od}(A) - 2a_{rs}^2 < \mathrm{Od}(A)$$

We have $b_{rs} = (a_{rr} - a_{ss})\sin\theta\cos\theta + a_{rs}(\cos^2\theta - \sin^2\theta)$. For $\theta = 0$, this is a_{rs}, and for $\theta = \frac{\pi}{2}$, this is $-a_{rs}$, so by the Intermediate Value Theorem, we can find some θ so that $\mathrm{Od}(U^T AU) < \mathrm{Od}(A)$. This concludes the proof. □

Proof of Theorem 2.2. We present the proof from [AZ18, Chapter 7].

Define the map $f_A : O(n) \to \mathbb{R}^{n\times n}$ by $f_A(P) = P^T AP$. Note that f_A is continuous, so since $O(n)$ is compact, the image $f_A(O(n))$ is compact too.

We know that the map $\mathrm{Od} : f_A(O(n)) \to \mathbb{R}$ is continuous, so it must attain a minimum on the set $f_A(O(n))$. Let the argument of the minimum be $D = Q^T AQ \in f_A(O(n))$. We now know that $\mathrm{Od}(D) = 0$, or otherwise by Lemma 2.5, there would exist some matrix M such that $\mathrm{Od}(M^T DM) < \mathrm{Od}(D)$, and $\mathrm{Od}(D)$ would not be the minimum. This completes the proof, since we have found a real orthogonal matrix Q such that $Q^T AQ$ is diagonal. □

As mentioned before, we can use Theorem 2.2 to prove the following bound:

Theorem 2.6 (Hadamard)**.** *For $n \times n$ matrices $A = (a_{ij})$ where $|a_{ij}| \le 1$ for all i, j, we have the upper bound*

$$|\det(A)| \le n^{\frac{n}{2}}.$$

To prove this theorem, we will first need a lemma that tells us that the maximum actually exists:

Lemma 2.7. *There is some $n \times n$ matrix A with entries in $[-1, 1]$ such that $\det(A)$ is maximal. Furthermore, A must have entries in $\{-1, 1\}$.*

Proof. We can view the determinant as a continuous function in the variables a_{ij}. Since the set of real $n \times n$ orthogonal matrices is compact, as we showed in Lemma 2.3, we know there is a maximum value of $\det(A)$. Note that the determinant is a linear function in each entry, so the maximum must be achieved by a matrix with entries in $\{-1, 1\}$. □

We now present the proof of Theorem 2.6 from [AZ18]:

Proof of Theorem 2.6. By Lemma 2.7, it suffices to prove the bound $\det(A) \le n^{\frac{n}{2}}$ for matrices with entries ± 1. The idea of the proof is to consider $B = (b_{ij}) = A^T A$ instead, which is a symmetric matrix, and then use the spectral theorem.

By the spectral theorem, there exists some real orthogonal matrix Q such that

$$Q^T BQ = Q^T A^T AQ = (AQ)^T (AQ) = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues. Notice that since the entries of B are only ± 1, $b_{ii} = a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2 = n$. Then, the sum of the diagonal $\mathrm{tr}(B) = \sum_{i=1}^n \lambda_i = \sum_{i=1}^n b_{ii} = n^2$.

We now bound $\det(B)$ using AM-GM:

$$\det(B) = \det\left((AT)^Q (AT)\right) = \prod_{i=1}^n \lambda_i \le \left(\frac{\sum_{i=1}^n \lambda_i}{n}\right)^n = \left(\frac{\mathrm{tr}(B)}{n}\right)^n = n^n$$

So now, we get the bound

$$|\det A| = \sqrt{\det B} \le n^{\frac{n}{2}}$$

as desired.

□

3. The Hadamard Conjecture

In this section, we explore the question of when the Hadamard bound is actually attainable.

Definition 3.1. Matrices with A with ± 1 entries that achieve $|\det(A)| = n^{\frac{n}{2}}$ are called **Hadamard matrices**.

Question 3.2. *For which n does a Hadamard $n \times n$ matrix exist?*

Example. For odd $n \geq 3$ that are not perfect squares, a Hadamard matrix clearly cannot exist, since the bound $n^{\frac{n}{2}}$ is not an integer.

Example. For $n = 4$, an example of a Hadamard matrix is

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Using expansion by minors, $\det(M)$ is equal to

$$\det \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} - \det \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - \det \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$= (2 - 0 + 2) - (-2 - 2 + 0) + (0 + 2 + 2) - (-2 + 0 - 2) = 4 + 4 + 4 + 4 = 16$$

which is the Hadamard bound $4^{\frac{4}{2}} = 16$.

To show a condition on when the Hadamard bound can be attained, we show the following:

Proposition 3.3. *The determinant of an $n \times n$ matrix $A = (a_{ij})$ with entries ± 1 is an integer multiple of 2^{n-1}.*

Proof. Our goal is to show that A can be transformed into an $(n-1) \times (n-1)$ matrix with entries -2 and 0, using the elementary row and column operations that don't change the absolute value of the determinant.

These operations include negating a row or a column, so note that we can first make all entries in the first row 1 by negating each column when necessary. We can then make all entries in the first column 1 by negating each row when necessary. Now, if we subtract the first row from every other row, we get 0's in the entries $a_{21}, a_{31}, \ldots, a_{n1}$.

Using expansion by minors, we can see that the determinant of the $n-1 \times n-1$ matrix A' formed by deleting the first row and column is equal to the determinant of A. This is a matrix consisting of only 0's and -2's. Each term in the determinant of A' is a product of $n-1$ entries, so dividing all entries by 2 has the effect of dividing the determinant by 2^{n-1}. The determinant of the resulting matrix k is an integer since the matrix only has entries in $\{-1, 0\}$, so $|\det(A)| = |\det(A')| = 2^{n-1}k$, as desired.

To show an example of this process, consider the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

We first make the first row and column all 1's:

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

We now subtract the first row from each of the other rows and then delete the first row and column:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & -2 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -2 & -2 \\ -2 & 0 & -2 \\ -2 & -2 & 0 \end{bmatrix}$$

If we divide all entries of this 3×3 matrix by 2 (to get another integer matrix), then we reduce the determinant of A by $2^3 = 8$, so we can confirm that $8 \mid \det(A)$. □

Using this property, we can show the following condition on when the Hadamard bound can be attained.

Corollary 3.4. *The Hadamard bound is unattainable except when $n = 1, 2$, or a multiple of 4.*

Proof. The case where n is odd and not a perfect square is obvious, since $n^{\frac{n}{2}}$ is not an integer. This leaves us with two cases to show:

Case 1. $n = (2k+1)^2$ for some $k \in \mathbb{Z}$. In this case, $n^{\frac{n}{2}}$ is clearly odd, and by Proposition 3.3, the determinant of any $n \times n$ matrix is an integer multiple of 2^{n-1}. This shows that the Hadamard bound is unattainable for these values of n.

Case 2. $n = 2(2k+1)$ for some $k \in \mathbb{Z}$. In this case, $n^{\frac{n}{2}} = 2^{2k+1}(2k+1)^{2k+1}$, so the largest power of 2 that divides $n^{\frac{n}{2}}$ is 2^{2k+1}. But by Proposition 3.3, the largest power of 2 that divides the determinant of any such matrix has to be at least $4k+1$. Thus, the Hadamard bound is unattainable for these values of n.

□

We can also ask about the other direction: if the size of the matrix n is a multiple of 4, is there necessarily a Hadamard matrix of size n? This is the Hadamard conjecture, which is still unsolved.

Conjecture 3.5. *Hadamard matrices exist for every multiple of 4.*

For small values of n, there have been constructions of Hadamard matrices:

Theorem 3.6. *For every multiple of 4 up to 1000 except 668, 716, and 892, a Hadamard $n \times n$ matrix exists.*

Proof. See [Djo07] for a summary of the progress so far on the Hadamard conjecture, as well as a recent construction of a Hadamard matrix of order 764. □

4. Other Bounds

Even if the Hadamard bound is unattainable for numbers that are not multiples of 4, we can still ask the question of how large these determinants can be for an $n \times n$ matrix with ± 1 entries. We have the following result:

Theorem 4.1. *For every n, there exists a $n \times n$ matrix with entries ± 1 whose determinant is greater than $\sqrt{n!}$.*

Proof. One idea is to compute the average of the determinants of the 2^{n^2} matrices and use this to show the existence of a matrix with determinant greater than $\sqrt{n!}$. However, it is not hard to see that the average is actually 0, so we instead consider the root mean square average

$$D_n = \sqrt{\frac{\sum_A (\det A)^2}{2^{n^2}}}$$

We want to show that $D_n^2 = \frac{\sum_A (\det A)^2}{2^{n^2}}$ evaluates to $n!$. We first expand out the sum using the definition of the determinant in terms of permutations:

$$D_n^2 = \frac{1}{2^{n^2}} \left(\sum_{A=(a_{ij})} \sum_{\pi} \operatorname{sgn}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} \right)^2$$

where the second sum ranges over the permutations of $1, 2, \ldots, n$, and $\operatorname{sgn}(\pi)$ denotes the sign of the matrix, i.e. 1 if the permutation is even and -1 is the permutation is odd. We can simplify this expression by rearranging the sums:

$$D_n^2 = \sum_{A=(a_{ij})} \sum_{\sigma} \sum_{\tau} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) a_{1\sigma(1)} a_{1\tau(1)} \cdots a_{n\sigma(n)} a_{n\tau(n)}$$

$$= \frac{1}{2^{n^2}} \sum_{\sigma} \sum_{\tau} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \left(\sum_{A=(a_{ij})} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} \right)$$

Notice that whenever $k = \sigma(i) \neq \tau(i)$ for some i, the terms with a_{ik} cancel each other out. This means that we only sum over the terms where $\sigma = \tau$, so $\operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)$ is always 1:

$$D_n^2 = \frac{1}{2^{n^2}} \sum_{\sigma} \sum_{a_{11}=\pm 1} \sum_{a_{12}=\pm 1} \cdots \sum_{a_{nn}=\pm 1} 1$$

$$= \frac{1}{2^{n^2}} \sum_{\sigma} 2^{n^2} = n!$$

Since the root mean square average $D_n = \sqrt{n!}$, we conclude that there exists an $n \times n$ matrix with entries ± 1 whose determinant is greater than $\sqrt{n!}$. □

Remark 4.2. Note that the bound given in Theorem 4.1 is actually pretty close to Hadamard's bound. By Stirling's approximation of the factorial,

$$\sqrt{n!} \approx \sqrt[4]{2\pi n} \left(\frac{n}{e} \right)^{\frac{n}{2}}$$

which is not far from $n^{\frac{n}{2}}$.

5. Bounds for Congruence Classes Mod 4

We can also show stronger bounds for other congruence classes $\pmod 4$. These bounds will actually look very similar to the Hadamard bound. We start off with the case of when n is odd:

Theorem 5.1 (Barba)**.** *Let A be an $n \times n$ matrix with entries ± 1. For n odd, we have the bound*

$$|\det(A)| \leq \sqrt{2n-1}(n-1)^{\frac{n-1}{2}}.$$

We first prove a lemma:

Lemma 5.2. *Let*

$$M_n = \begin{bmatrix} a & a_{12} & \cdots & a_{1n} \\ a_{21} & m & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & m \end{bmatrix}$$

Then for any a satisfying

$$0 \leq a \leq \min_{2 \leq j \leq n} |a_{1j}|$$

we have $\det(M_n) \leq a(m-a)^{n-1}$.

Proof. We use a well-known version of Hadamard's inequality: that for a real positive definite symmetric matrix, the determinant is less than or equal to the product of the diagonal (See [Ost52] for more details on this). Our goal to use elementary row and column operations to make the 2nd through nth diagonal elements into something close to $m-a$.

The first operation we do is subtract the first row multiplied by a_{1i}/a from the ith row, so our matrix is now

$$\begin{bmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & m - \frac{a_{12}^2}{a} & \cdots & a_{2n} - \frac{a_{12}a_{1n}}{a} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} - \frac{a_{12}a_{1n}}{a} & \cdots & m - \frac{a_{1n}^2}{a} \end{bmatrix}.$$

Notice that for $2 \leq i, j \leq n$,

$$a_{ij} = \begin{cases} a_{ij} - \frac{a_{1j}a_{1i}}{a} & i \neq j \\ m - \frac{a_{1i}^2}{a} & i = j \end{cases}$$

which means that the matrix is symmetric except for the first row and column. To make M_n symmetric, we subtract the first column multiplied by a_{1i}/a from the ith column, so that our matrix is now

$$\begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & m - \frac{a_{12}^2}{a} & \cdots & a_{2n} - \frac{a_{12}a_{1n}}{a} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} - \frac{a_{12}a_{1n}}{a} & \cdots & m - \frac{a_{1n}^2}{a} \end{bmatrix}.$$

and the rest of the entries are unchanged. This new matrix has determinant equal to $\det(M_n)$, so we can use the fact that $\det(M_n)$ is less than or equal to the product of the diagonal entries. Then, for any a satisfying $0 \leq a \leq \min_{2 \leq j \leq n} |a_{1j}|$, we have:

$$\det(M_n) = a \prod_{i=2}^{n} \left(m - \frac{a_{1i}^2}{a} \right) \leq a \prod_{i=2}^{n} (m-a) = a(m-a)^{n-1}$$

as desired. ∎

We now present the proof of Theorem 5.1 from [Woj64]:

Proof of Theorem 5.1. We will actually prove a stronger claim; let $A_n = (a_{ij})$ be an $n \times n$ positive definite symmetric matrix with diagonal elements equal to m. Then for any a satisfying

$$0 \leq a \leq \min_{1 \leq i,j \leq n} |a_{ij}|$$

we have $\det(A_n) \leq (m + na - a)(m - a)^{n-1}$.

We want to show this using induction, which suggests that we should try expansion by minors. For the base case, we just have the one-dimensional matrix $\begin{bmatrix} m \end{bmatrix}$. The inequality becoems $\det(A_1) \leq (m + a - a)(m - a)^0 = m$, which clearly holds.

Now we assume our claim for some $n - 1 \geq 1$, and we let

$$A_n = \begin{bmatrix} m & a_{12} & \cdots a_{1n} & \\ a_{21} & m & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & m \end{bmatrix}$$

Note that when we expand by minors, the first term will be

$$m \begin{bmatrix} m & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & m \end{bmatrix}$$

so we can write $\det(A_n)$ as

$$\det(A_n) = \det \begin{bmatrix} a & a_{12} & \cdots a_{1n} & \\ a_{21} & m & \cdots a_{2n} & \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & m \end{bmatrix} + \det \begin{bmatrix} m - a & 0 & \cdots & 0 \\ a_{21} & m & \cdots a_{2n} & \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & m \end{bmatrix}$$

Let A'_n be the matrix in the first term. Then $\det(A_n) = \det(A'_n) + (m - a)\det(A_{n-1})$. We now consider two cases:

Case 1. $\det(A'_n) > 0$. The smaller minors of A'_n are the minors of A_n, so A'_n is positive definite. This means that we can apply Lemma 5.2, and we have the bound $\det(A'_n) \leq a(m - a)^{n-1}$. So $\det(A_n) \leq a(m - a)^{n-1} + (m - a)\det(A_{n-1})$

$$\leq a(m - a)^{n-1} + (m - a)(m + (n - 1)a - a)(m - a)^{n-2}$$

$$= a(m - a)^{n-1} + (m + n - 2a)(m - a)^{n-1} = (m + na - a)(m - a)^{n-1}$$

as desired.

Case 2. $\det(A'_n) \leq 0$. In this case, we can bound $\det(A_n)$ as

$$\det(A_n) \leq (m - a)A_{n-1} \leq a(m - a)^{n-1} + (m - a)\det(A_{n-1})$$

so the inequality from the previous case holds as well.

This completes the proof of our claim that for any positive definite symmetric matrix with diagonal elements equal to m and a satisfying

$$0 \leq a \leq \min_{1 \leq i,j \leq n} |a_{ij}|$$

we have $\det(A_n) \leq (m + na - a)(m - a)^{n-1}$.

Now, notice that if we have an $n \times n$ matrix A with entries ± 1. Consider the matrix $B = AA^T$. The ijth entry of B is the dot product of the ith row and the jth column, which is a sum of n terms that are each either 1 or -1. If n is odd, then this cannot be 0, so $1 \leq |b_{ij}|$ for all $1 \leq i, j \leq n$. The diagonal terms b_{ii} are all n, so if we apply our claim, then we get the bound

$$\det(B) = \det(A)^2 = (2n - 1)(n - 1)^{n-1}$$

Taking the square root of both sides gives us the desired inequality. □

Proposition 5.3. *Barba's bound is unattainable unless $2n - 1$ is a perfect square.*

Proof. This clearly follows from the $\sqrt{2n - 1}$ factor in the bound. □

In particular, we can show the following:

Corollary 5.4. *Barba's bound is never attainable when $n \equiv 3 \pmod 4$.*

Proof. Suppose $n = 4k + 3$. Then, $2n - 1 = 2(4k + 3) - 1 = 8k + 5$. Since perfect squares cannot be congruent to 5 (mod 8), Proposition 5.3 tells us that Barba's bound is never attainable in this case. □

For the other direction, we have the following result:

Theorem 5.5. *Let $n = 2(q^2 + q) + 1$, where q is an odd prime power. Then, there is an $n \times n$ matrix with entries ± 1 that attains the bound in Theorem 5.1.*

Proof. The proof is out of the scope of this paper; see [NR97, Proposition A] for details. □

Theorem 5.6 (Ehlich–Wojtas)**.** *Let A be an $n \times n$ matrix with entries ± 1. For $n \equiv 2 \pmod 4$, we have the bound*

$$|\det(A)| \leq (2n - 2)(n - 2)^{\frac{n-2}{2}}.$$

To prove this theorem, we first need a lemma:

Lemma 5.7. *Let M be a symmetric $n \times n$ matrix with all diagonal elements equal to m satisfying*

$$a_{ik} = a_{jk} = 0 \implies a_{ij} \neq 0$$

and let a be a real number satisfying

$$0 \leq a \leq \min_{a_{ij} \neq 0} |a_{ij}|.$$

Then,

$$|\det(M)| \leq (m + pa - a)(m - pa - a + na)(m - a)^{n-2}$$

where p is the maximum number of zeroes in a row of M.

Proof. We refer the reader to [Woj64, Lemma 2] for the proof. □

We now present the proof of Theorem 5.6 from [Woj64]:

Proof of Theorem 5.6. Let $B = A^T A$; note that as mentioned before, B is a symmetric $n \times n$ matrix with diagonal elements all equal to n. Since n is even, all nonzero elements of B are greater than or equal to 2 in absolute value, so we can take $a = 2$ in the statement of Lemma 5.7.

To find an upper bound for $\det(A)$, we take the derivative of the bound in Lemma 5.7 and set it to 0:

$$\frac{\mathrm{d}}{\mathrm{d}\,p}(m+pa-a)(m-pa-a+na)(m-a)^{n-2}=0$$

Using the product rule,

$$(m-pa-a+na)(a)+(m+pa-a)(-a)=0$$
$$m-pa-a+na-m-pa+a=0$$
$$2p-n=0$$

so the bound achieves a maximum at $p=\frac{n}{2}$. Setting $p=\frac{n}{2}$, $m=n$, and $a=2$ gives us the bound

$$\det(B)=\det(A)^2=(2n-2)(n-2)^{n-2}$$

Square rooting both sides gives us the desired result. □

Ehlich also showed the following about this bound in [Ehl64]:

Proposition 5.8. *The Ehlich–Wojtas bound is unattainable unless* $2n-2$ *can be written as a sum of two squares.*

We can also ask about when the bound in Theorem 5.6 is actually attainable. We have the following result:

Theorem 5.9. *Let* $n=4(q^2+q)+2$*, where* q *is an odd prime power. Then, there is an* $n\times n$ *matrix with entries* ±1 *that attains the bound in Theorem 5.6.*

Proof. The proof is out of the scope of this paper; see [NR97, Theorem B] for details. □

Remark 5.10. The $n\equiv 3 \pmod 4$ case is more complicated; see [Tam06] for a recent result on bounds for $n\times n$ matrices with entries ±1 for $n\equiv 3 \pmod 4$.

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