Chromatic Number of the Plane

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1 Introduction

The chromatic number of the plane problem(also known as Hadwiger Nelson problem) asks for the number of colors needed to color the plane such that no two points at unit distance have the same color. This problem has been around since the past seven decades and has baffled many mathematicians. In this paper we shall first try to understand the problem better and look at how we can give a bound to the chromatic number of the plane by suitable bounds. We present several proofs of the same involving unit distance graphs like the Mosers' Spindle and Golomb Graph. For the reader looking for an intuitive proof we present an alternate construction. Finally in the end we briefly look at the recent advancement made in this problem by Aubrey de Grey using certain computer algorithms. The problem is still open as the chromatic number of the plane can be either $5, 6$ or 7.

2 Hadwiger Nelson Problem

The following problem from the 3rd Colorado Mathematical Olympiad [1] serves as a good appetizer to the main problem:
Santa Claus and his elves paint the plane in two colors, red and green. Prove that the plane contains two points of the same color exactly one mile apart.

Proof. Toss on the plane an equilateral triangle with side lengths equal to one mile. Since its three vertices (pigeons) are painted in two colors (pigeonholes), there are two vertices painted in the same color (at least two pigeons in a hole). These two vertices are one mile apart. □

The following generalization of the above problem is straight forward:

Question 2.1. *How many colors are needed to color the plane so that no two points at unit distance are the same color?*

This number is called the chromatic number of the plane and is often denoted by χ. A segment here will stand simply for a pair of points. A segment in a colored plane is called monochromatic if both its endpoints are colored in the same color.

3 Lower and Upper Bounds

In this section we discuss the proofs of the upper and lower bounds of χ.

3.1 Lower Bound

Theorem 3.1. *$\chi \geq 4$, i.e., any 3-colored plane contains a monochromatic segment of length 1.*

Proof 1. Toss on the given 3-colored plane what we now call the Mosers' Spindle (Figure 1). Every edge in the spindle has length 1. Assume that the seven vertices of the spindle do not contain a monochromatic segment of length 1. Call the colors used to color the plane red, white, and blue. Let the point A be red, then B and C must be one white and one blue, therefore, D is red. Similarly E and F must be one white and one blue, therefore, G is red. We now have a monochromatic segment DG of length 1 in contradiction to our assumption. □

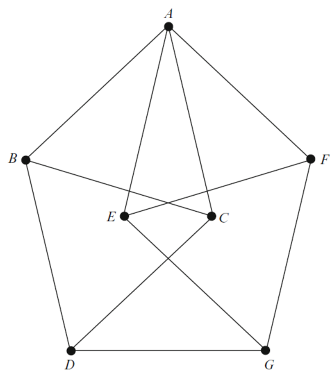

Figure 1: Mosers' Spindle

Proof 2. Just toss the Golomb Graph on a 3-colored (red, white and blue) plane (Figure 2). Assume that in the graph there are no adjacent (i.e., connected by an edge) vertices of the same color. Let the center point be colored red, then since it is connected by unit edges to all vertices of the regular hexagon H, H must be colored white and blue in alternating fashion. All vertices of the equilateral triangle T are connected by unit edges to the three vertices of H of the same color, say, white. Thus, white cannot be used in coloring T , and thus T is colored red and blue, which implies that two of the vertices of T are assigned the same color. This contradiction proves that 3 colors are not enough to properly color the ten vertices of the Golomb graph, let alone the whole plane. □

The above proofs are simple and beautiful but how does one come up with such a construction? We present another proof of the above theorem which is less elegant but more naturally found.

Proof 3. Assume that a 3-colored plane does not contain a monochromatic segment of length 1. Then an equilateral triangle ABC of side 1 will have one vertex of each color (Figure 3). Let A be red. The point A' symmetric to A with respect to the side BC must be red as well. If we rotate our rhombus ACA'B about A through any angle, the vertex A' will have to remain red due to the

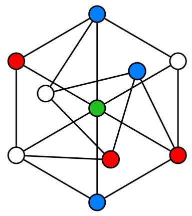

Figure 2: Golomb Graph

same argument as above. Thus we get a whole red circle of radius AA' (Figure 3). Surely it contains a cord d of length 1, in contradiction to our assumption. $\square$

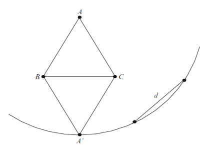

Figure 3: Alternate Construction [2]

3.2 Upper Bound

Theorem 3.2. *$\chi \leq 7$, i.e., there is a 7-coloring of the plane that does not contain two points of the same color distance 1 apart.*

Proof. We can tile the plane by regular hexagons of side 1. Now we color one hexagon in color 1, and its six neighbors in colors $2, 3, \ldots 7$ (Figure 4). The union of these seven hexagons forms a symmetric polygon P of 18 sides. Translates of P (i.e., images of P under translations) tile the plane and determine how we color the plane in 7 colors. It is easy to prove that no color has monochromatic segments of any length d, where

$$2 < d < \sqrt{7}$$

Thus, if we shrink all linear sizes by a factor of, say, 2.1, we will get a 7-coloring of the plane that has no monochromatic segments of length 1. $\square$

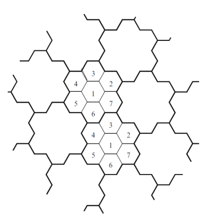

Figure 4: Hexagonal Tiling

4 Aubrey de Grey's New Lower Bound

In 2018, computer scientist and biologist Aubrey de Grey found a 1581-vertex[3], non-4-colorable unit-distance graph. The proof is computer assisted and hence we only give an outline of the construction.

Theorem 4.1. $\chi \geq 5$ *i.e., any* 4*-colored plane contains a monochromatic segment of length 1.*

(1) We note that the 7 -vertex, 12-edge unit-distance graph H consisting of the centre and vertices of a regular hexagon of side-length 1 can be colored with at most four colors in four essentially distinct ways (that is, up to rotation, reflection and color transposition). Two of these colorings contain a monochromatic triple of vertices and two do not.

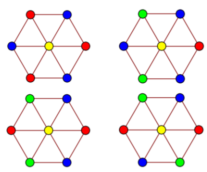

Figure 5: The essentially distinct ways to color H with at most four colors.

(2) We construct a unit-distance graph L that contains 52 copies of H and show that, in all 4 -colorings of L, at least one copy of H contains a monochromatic triple.

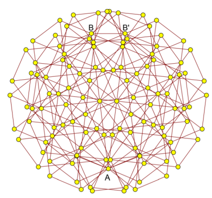

Figure 6: The graph L, containing 121 vertices and 52 copies of H.

(3) We construct a unit-distance graph M that contains a copy of H and show that there is no 4 -coloring of M in which that H contains a monochromatic triple. Thus, the unit-distance graph N created by arranging 52 copies of M so that their counterparts of H form a copy of L is not 4 -colorable. This completes our demonstration that the χ is at least 5 .

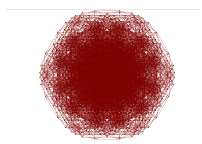

Figure 7: The graph M

References

[1] Alexander Soifer. *The Colorado Mathematical Olympiad and Further Explorations*. Springer, 2011.

[2] Alexander Soifer. *The Mathematical Coloring Book*. Springer, 2009.

[3] Aubrey D. N. J. de Grey. The chromatic number of the plane is at least 5, 2018.