Boolean Algebra : Stone's Representation Theorem.

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1 Introduction

In this article, we discuss primarily What is *Boolean Algebra*?, Fundamental laws of Boolean Algebra, and Stone's Representation theorem.

2 What is Boolean Algebra ?

T he topic Boolean algebra is a branch of algebra first introduced by George Boole that involves mathematical logic. By a binary operation on a set S we mean a way of combining an ordered pair of elements of S to give a uniquely defined element of binary operations in arithmetic for example:Closure property, Unary operation,etc.

Suppose B is a set containing two distinct special elements called 0 and 1;suppose there are two binary operation defined on B,which we shall denote by $+$ and $\times$and a *unary* operation denoted by $'$.Then B a Boolean Algebra if the following five pair of laws hold for all members x,y and z in B.

2.1 Some Fundamental Laws..

1. Commutative laws
a.$X + Y = Y + X$
b.$X \times Y = Y \times X$
2. Associative laws
a.$(X + Y) + Z = X + (Y + Z)$
b.$(X \times Y) \times Z = X \times (Y \times Z)$
3. Distributive laws
a.$X + (Y \times Z) = (X + Y) \times (Y + Z)$
b.$X \times (Y + Z) = (X \times Y) + (X \times Z)$
4.Identity laws
a.$X + 0/X$
b.$X \times 1 = X$
5. complement laws

a.$X + X' = 1$ b.$X \times X' = 0$
We shall call it $'Axioms'$ of Boolean algebra.

2.2 Some Important Results in Boolean Algebra

•**Principle of duality**: The dual of every theorem in Boolean Algebra is also a theorem.
•**(Further) Distributive law**: In any Boolean algebra all elements X,Y,and Z satisfy,
a.$(x + y)z = xz + yz$
b.$xy + z = (x + z)(y + z)$
•**Idempotent law**:In any Boolean algebra ,every element x satisfies, a. $x + x = x$ and b.$xx = x$.
•**Absorption law**:In any Boolean algebra all elements x and y satisfies , a. $x + xy = x$ and b.

$$x \times (x + y) = x$$

•**Nullity law**: In any Boolean algebra every element x satisfies , $x + 1 = 1$ and b.$x \times 0 = 0$
•**Uniqueness of complement** : If x and y are any elements of any Boolean algebra and if $x + y = 0$ and $xy = 0$ then,$y = x'$.
•**Involution law** : $(x')' = x$.
• 0'=1 and 1'=0.
•**de Morgan's law** :In any Boolean algebra,all elements x and y satisfies, a.$(x + y)' = x'y'$ b.$(xy)' = x' + y'$.

Now that we've established Fundamental of Boolean algebra,we can begin exploring Stone's Representation theorem.

3 Stone's Representation Theorem

The study of Boolean algebra,has a big impact in variety of fields in mathematics. From functional analysis to other part of algebra itself , Stone's research along with that of his colleagues working on the same field had a lot of impact. So let's discuss Stone's Representation theorem...

Before going to Stone's Representation theorem we'll go for $Ultrafilters$ and $Atoms$, which are foundational elements of Boolean algebras.

3.1 Ultrafilters

Definition 3.1: A subset p of a Boolean algebra A is a $filter$ if (a)$1 \in p$,(b)if $x, y \in p$ then $x \cdot y \in p$ (c).if $x \in p$,$y \in A$ and $x \leq y$ then $y \in p$. For $a \in A$,p is a principal filter if $p = x \in X|a \leq x$.we say p is the *principal filter generated by a*.If $p = 1$,then p is a *trivial filter* and if $0 \notin p$,then p is *proper filter*.

Definition 3.2:An $Ultrafilter$ is a filter p of a Boolean algebra A where $x \in p$ or $-x \in p$, but not both, for each $x \in A$ If p is proper and $x + y \in p$ then either $x \in p$ or $y \in p$ for $x, y \in A$,then p is a *prime filter*. If p is proper and there is no proper filter q of A with $p \subset q$, then p is a *maximal filter*.

The set of Ultrafilters of A is $UltA = \{p \subseteq A|$p is an ultrafilter of A$\}$

Definition 3.3:The *Stone* map of Boolean algebra A is a map $s \to p(UltA)$ where $s(x)$=$p \in UltA|x \in p$

Definition 3.4: If ϕ:$A \to B$ is a Boolean algebra homomorphism and $\sum^A M$ exists for $M \subseteq A$ then

$$\phi$$

preserves $\sum^A M$ if $\sum^B$

$$\phi[M]$$

exists an $\phi(\sum^A) = \sum^B \phi(M)$ similarly ϕ preserves $\Pi^A M$ if $\phi(\Pi^A M)$=$\Pi^B \phi[M]$.For an Ultrafilter p of A,p preserves $\sum M$ if for some $m \in M, \sum M \in p \Longrightarrow m \in p$ and preserves ΠM if $M \subseteq p \Longrightarrow \Pi M \in p$

3.2 Atoms

Definition 3.5: If B is Boolean algebra and $a < b$ then $a \in b$ is an *Atom* if $0 < a$ and there does not exists $x \in B$ such that $0 < x < a$.The set of atoms of B is denoted by *At B*. We say B is atomless if it contains no atom and atomic if there exists an atom such that $a \leq x$,for each positive $x \in B$

Lemma 3.1 :If B is a finite Boolean algebra and b is any nonzero element in B,then there exists an atom a in B such that $a \leq b$.

proof: If b is an atom in B ,then a=b.

suppose, b is not an atom in B. Then there exists $a_1 \in B$ such that $0 < a_1 < b$. If a_1 is atom in B then $a_1 = a$ and if it is not an atom in B then there exists a_3. If a_3 is an atom in B then $a_3 = a$ but if not,then we continue in this manner and eventually obtain $0 < a_n < ... < a_3 < a_2 < a_1 < a < b$ since B is finite,$a_n < b$.

Lemma 3.2 :If $b, c \in B$ and $b \nleq B$,then there exists an atom $a \in B$ such that $a \leq b$ and $a \nleq c$.

proof: Since $b \nleq c$ then $b > c$ and $b \wedge c'$.Then by *lemma*3.1,there exists $a \in B$ such that $a \leq b \wedge c'$.By definition of glb $a \leq b$ and $a \leq c'$.Now if $a \leq c$ and $a \leq c'$,then $a \leq c \wedge c'$ but $c \wedge c' = 0$ and so $a \leq 0$ this contradicts the fact that a is an atom in B. Therefore $a \nleq c$.

Lemma 3.3:Suppose $b \in B$ and $a_1, a_2, ..a_n$ are all atoms in B such that $b = a_1 \vee a_2 \vee a_3 \vee ... \vee a_n$. If a is an atom in B and $a \leq b$ then $a = a_i$, for some $i = 1, 2, .., n$.

Proof:Since $a \leq b$, then $a \wedge b = a$ so,$a = a \wedge b = a \wedge (a_1 \vee a_2 \vee ... \vee a_n) = (a \wedge a_1) \vee (a \wedge a_2) \vee ... \vee (a \wedge a_n)$. Now since a is an atom in B,then $a \leq 0$. so for some $i = 1, 2, ..., n, a \wedge a_1 \leq 0$ Therefore,$a = a_i$.

Lemma 3.4:If $b \in B$ and $a_1, a_2, ..., a_n$ are all of the Atoms that satisfies $a_i \leq b(i = 1, 2, ..., n)$ in B,then $b = a_1 \vee a_2 \vee ... \vee a_n$.

Proof: We will let $a_1 \vee a_2 \vee ... \vee a_n = c$.Since each $a_i \leq b$,then $c \leq b$.Now suppose $b \nleq c$.Then by lemma 3.2,there exists an atom in B such that $a \leq b$ and $a \nleq c$.But since $a \leq b$ and a is an atom in B , then $a = a_i$, for some

$i = 1, 2, .., n$.Therefore $a \leq c$, contradicting $a \not\leq c$,so $b \leq c$.Then $c \leq b$ and $b \leq c$,implying $b = c$. Therefore $b = a_1 \vee a_2 \vee ... \vee a_n$.

Definition-3.6:Let A and B be the Boolean algebras.A Boolean $Homomorphism$ is a mapping $f : A \to B$ such that all $p, q \in A$

$$1.f(p \wedge q) = f(p) \wedge f(q)$$

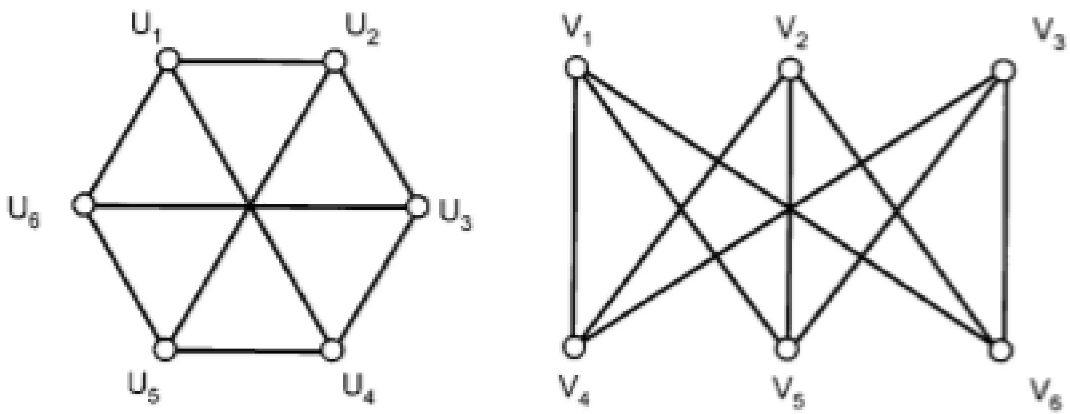
.

$$2.f(p \vee q) = f(p) \vee f(q)$$

and

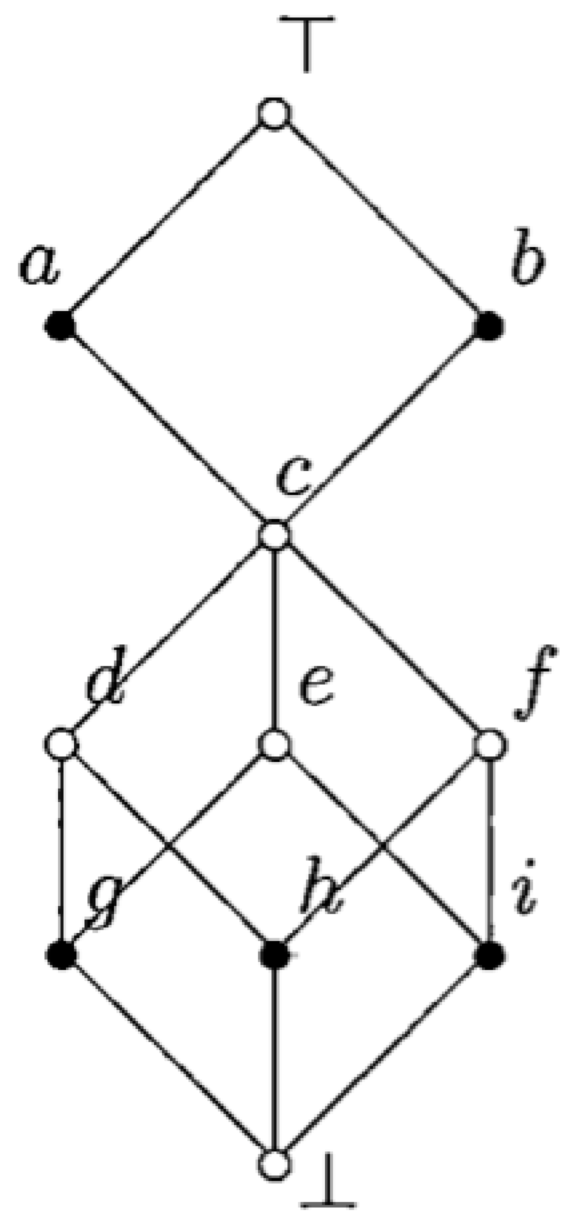
$$3.f(-a) = -f(a)$$

The surjective (onto) Homomorphism is called an $Epimorphism$. The injective (one-to-one) Homomorphism is called a *monomorphism*. A Homomorphism that is both surjective and injective is called an $Isomorphism$. If there exists $Isomorphism$ between A and B,then they are $Isomorphic$.



From the above figures,,we can get the idea of isomorphism.

THEOREM 3.2:*If B is a finite Boolean algebra ,then B is isomorphic to the power set $P(S)$ of some nonempty finite set S.This is* ***Stone's Representation Theorem for finite Boolean algebras***



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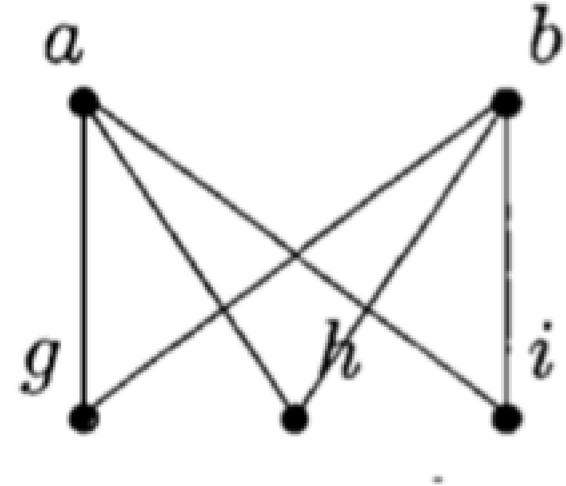

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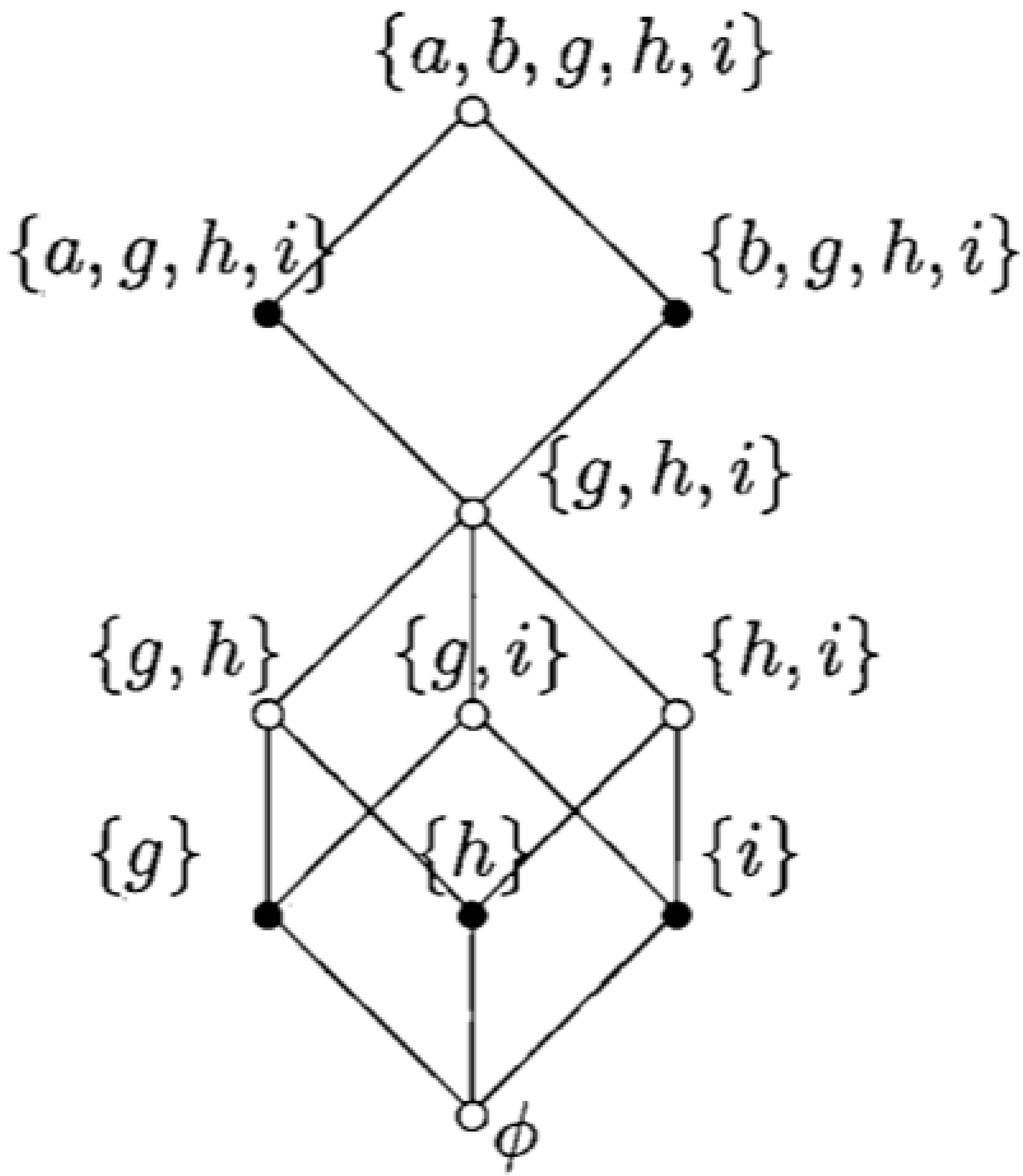


$H(J(L))$

(from this figures we can understand the Stone's Representation theorem geometrically.)

Now let's prove the theorem using definitions and lemma which we mentioned earlier.

Proof: Let B be a finite Boolean algebra and let S be a set of all atoms in B.e want to show that B is isomorphic to the power set of S ($B \cong P(S)$);that is we want to show that,there exists a map $\phi : B \to P(S)$ such that ϕ is a bijection and ϕ is order preserving

Since $B \neq$ there exists $b \in B$ We will define $\phi : B \to P(S)$ such that for $b \in B$,then $\phi(b) = A$,where $A = a|a \in Sanda \leq b$.If b=0 then $\phi(b) =$,Since there does not exist $a \in S$ such that $a \leq 0$ (by the definition of atom).Also,if $b = 1$,then $\phi(b) = S$,since for all $a \in S, a \leq 1$.We will assume $b \neq 0, b \neq 1$

First we will consider $a \in S|a \leq b$ for some $b \in B$=$a_1, a_2, ...a_k) \subseteq S$.Then $b = a_1 \vee a_2 \vee a_3 \vee ... \vee a_k$ for any atom $a \leq b$ in B,then $a = a_i$ for some $i = 1, 2, ..., k$ Therefore $\phi(b) = \phi(a_1 \vee a_2 \vee ... \vee a_k) = a_1, a_2, a_3, ..., a_k$ and so ϕ is onto.

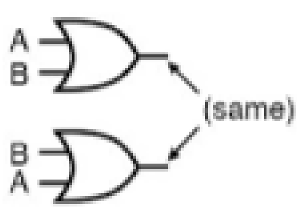
Now,ϕ is order preserving if for $b, c \in B, b \leq c$ in B if and only if $\phi(b) \subseteq \phi(c)$ in $P(S)$.If $b \leq c$ then every atom a such that,$a \leq b$,is an atom such that $a \leq c$ therefore, $\phi(b) \subseteq \phi(c)$. Now assume that $b \not\leq c$.Then there exists an atom a such that $a \leq b$ and $a \not\leq c$.Then $a \in \phi(b)$ and $a \notin c$.This implies $\phi(b) \not\subseteq \phi(c)$

Lastly,if $\phi(b) = \phi(c)$,then $\phi(b) \subseteq \phi(c)$ and $\phi(c) \subseteq \phi(b)$, Since ϕ is order preserving ,$\phi(b) \leq \phi(c)$ implies $b \leq c$ and similarly $\phi(c) \leq \phi(b)$ implies $c \leq b$.Then by anti symmetry,$b = c$ and therefore ϕ is one-to-one.

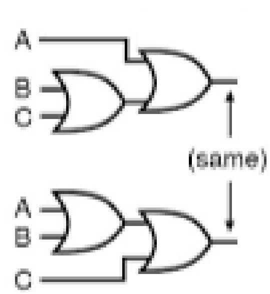
4 More about Boolean algebra...

Boolean algebra is useful in field of electrical engineering .In particular, by taking variables to represent value of On and Off (or 0 and 1) Boolean algebra is used to design and analyze digital switching circuitry, such as found in personal computers, pocket calculators, CD player, cellular telephones, and the host of other electronic products.

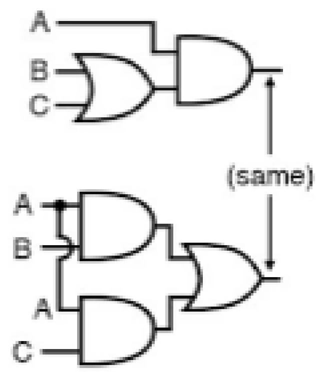
We can see this with the help of applications of Axioms (or Fundamental rules) of Boolean algebra,like



Commutative Property.



Associative Property.



Distributive Property.

5 Acknowledgment

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•**References**

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Thanks