Decomposition of Lines in the Plane and the Slope Problem

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1 Introduction

Take a set of points. Assuming that not all points are on the same line, what can you say about the lines passing through said points? James Joseph Sylvester posed a conjecture. He was probably inspired by a related problem in algebraic geometry. This conjecture was perhaps the most famous problem on the configuration of lines. Incidentally proving the conjecture also solved the related algebraic geometry problem.

The conjecture was proved by Eberhard Melchior in 1941. Unaware of that proof, Tibor Gallai proved the theorem again. The theorem is now named after him: Sylvester - Gallai theorem.

In this paper, we will look at the Sylvester - Gallai theorem as well as a few generalizations and related problems. One such related problem is the slope problem posed by P.R Scott. In this problem we look at the number of slopes formed by a set of points, where not all of the points are collinear.

2 Background

Let us begin by defining a few necessary concepts that we will be using in these theorems. First, we need a definition of the number of slopes in our configuration:

Definition 1. The number of slopes determined by a configuration of points is the number of lines of different slopes determined by said configuration.

This means that when two lines are parallel, they only create one new slope. Most theorems will not need any more definitions; however, we also will look at a related theorem in graph theory and need to define a graph:

Definition 2. A graph is set of vertices V and a set E of pairs of elements of V.

Finally, we will be looking at a few very specific types of graphs. We will need to define a complete graph and a complete bipartite graph:

Definition 3. A complete graph is a graph where the set E contains all possible pairs of elements of V.

Definition 4. A complete bipartite graph is a graph such that the set V has two subset A and B such that every element of A has an edge to every element of B and vice versa.

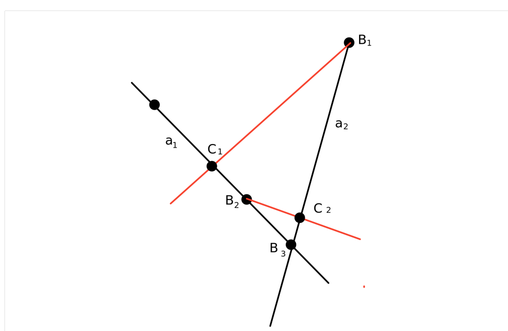
We will also be assuming the metric axioms and order axioms.

3 Main Results

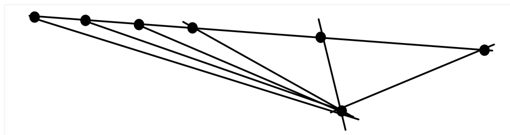
Let us begin by proving that there is at least one line passing through exactly two points. This is, in fact, the Sylvester - Gallai theorem and will be a stepping stone for the rest of the problems.

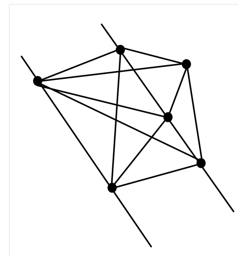
Theorem 1 (Sylvester - Gallai). Given any configuration of n points not all on the same line, then there is at least one line with exactly two points.

Proof. We will construct the needed line and then proceed to prove that it passes through exactly two points. The theorem is clearly true for three points. Let B be the set of $n > 3$ points such that not all of them are on the same line, and A be the set of lines passing through said points. Let C be the set of points not in B. Let us choose a line a_1 in A and some point B_1 in B such that the distance between a_1 and B_1 is the smallest out of any line and point pair. Let C_1 be the closest point to B_1 on line a_1. Note that C_1 may be in C. Assume toward a contradiction that the line a_1 contains more than two points in B. That would mean that there are two points on line a_1, where the two points are on the same side of C_1. Let us call them B_2 and B_3 where the distance between B_2 and C_1 is shorter than the distance between B_3 and C_1. Let the line $a_2 \in A$ contain the points B_1 and B_3. Let C_2 be the closest point to B_2 on line a_2. Note that C_2 may be in C. The distance between B_2 and C_2 must be shorter than the distance between B_1 and C_1, which is a contradiction. QED



There are two logical next steps. The first is to generalize and see if this result is true for more than just lines and can apply to general incidence geometries. The other is to wonder if there are more lines passing through exactly two points. We will look at both. Let us begin by looking at whether there are more lines passing through exactly two points. Let us look at a few examples:





We can see in these examples that there is a lot more than one line passing through exactly two points. In fact, there are always at least n lines.

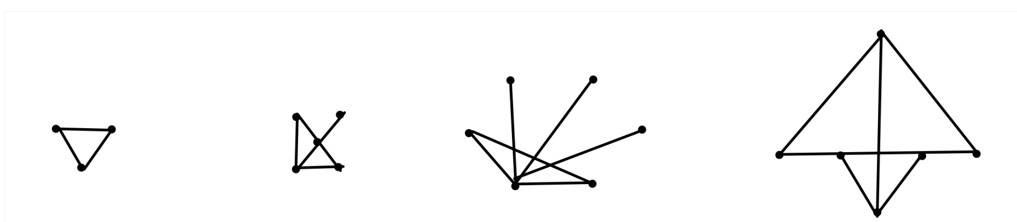
Theorem 2. Given any set P of n points, not all on the same line, then there is at least n lines with exactly two points.

Proof. We will use a proof by induction.

Base case: It is true when $n = 3$.

Inductive step: Assume the theorem is true when $n = x$. Let L be the set of lines passing through points in our configuration. By Theorem 1, for a configuration with $x+1$ points there must be at least one line a passing through exactly two points b and c. Take the sets $A = P/\{b\}$ and $B = P/\{c\}$. As the points in set P do not all lie on the same line, the points in A or the points in B do not all lie on the same line. Therefore, by our inductive hypothesis there are at least $x + 1$ lines in L. QED

Now before we look at a generalization of the Sylvester - Gallai theorem, let us take a quick detour and look at the slope problem. Instead of counting the number of lines, let us look at the number of slopes a configuration of n points contains. Let us again look at a few examples:



In these examples we can notice that there is always at least $n - 1$ slopes.

Theorem 3. Given a set P of n points not all on the same line, such that $n \geq 3$, then there are at least $n - 1$ slopes determined by P.

Proof. We will prove the theorem for an even number of vertices by setting up a sequence of permutations of vertices and looking at the length of the sequence. Looking back at our examples we notice that when there is an even number of points in P, the configuration of points determines at least n slopes. It suffices to prove this to prove the rest of the theorem. This is because for any configuration of $n = 2m + 1$ points, we can find a subset of $n - 1$ points that already contains $n-1$ slopes, and when $n=3$ the theorem is obvious. Let's take a configuration of $2m$ points with a slopes such that $a > 2$. Let us construct a combinatorial model. Let us look at a one dimensional projection. Let the direction of this one dimensional projection not be a slope in this configuration. Let us number the points in the order they appear in a one dimensional projection. Let us call this permutation $\pi_0 = 123...n$. Let us change the original direction by rotating it counter-clockwise. A rotation of $180°$ yields the permutation: $\pi_0 \to \pi_1 \to \pi_2 \to ... \to \pi_a$.

We can notice a lot about this sequence. First, a is the number of slopes in the configuration. Second, $\pi_a = n...j...i...321$. Third, for any i and j where $i < j$, as the sequence progresses, i and j get swapped exactly once. Finally, every move consists of one are more swaps.

Now all we must prove is that $a > n$. You may wonder why we decided to look at the case when n is even. We did this so that now we can split each permutation in half. Let us look at the move $\pi_i \to \pi_{i+1}$. If this move affects both sides, it is called a crossing move. This crossing move has order b if it moves $2b$ letters from one side to the other side. Therefore, if each move has order: $b_1, b_2...b_c$, the amount of letters that switch sides is:

$$\sum_{n=1}^{c} 2b_i > n-1$$

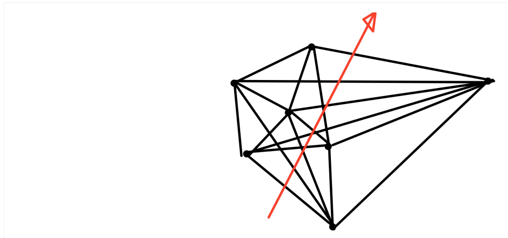
We know that $2b_i = n$ only if $a = 1$. Therefore, there must be at least two crossing moves. If the move: $\pi_i \to \pi_{i+1}$ does not affect both sides, it is called a touching move or an ordinary move. A touching move is when the swap touches the barrier between the two sides. All other moves are ordinary moves. We can denote every move with a letter: C, T or O. We know that between any two C moves there is a T move, and between a C move of order b and a T move there are at least $b-1$ O moves. Therefore, the sequence must also satisfy: between a T move and a C move of order b there are at least $b-1$ O moves. We can now complete the proof. By our rules if the direction rotates indefinitely, then the the pattern must be: $T, O, O...O, C, O, O..., O$. The number of swaps must be $1+(a-1)+1+(a-1)$. Taking a segment of length a, we get that the amount of letters that switch sides is:

$$a+1 > \sum_{n=1}^{c} 2b_i > n-1$$

Thus, $a > n-1$. QED

Now we can generalize the Sylvester - Gallai theorem. The following theorem was first proved by John Conway. The most beautiful proofs are the ones that are one liners, then come the proofs that are almost one liners. This is one such proof:

Theorem 4. Given a set S such that S has $n > 3$ elements, if A_1, A_2, A_3 ... A_m be subsets of S such that for any two elements $a, b \in S$ only one A_i contains both a and b, then $m \geq n$.



Proof. We will use a combinatorial proof. Assume toward a contradiction that $m < n$. For $a \in S$ let x_a be the number of sets A_i containing a. If there is some j such that $a \notin A_j$ then $a_x \geq |A_j|$. This is because the $|A_j|$ sets containing x and an element of A_j are distinct. By our assumptions $|A_j|m < na_x$. Thus, $(n - |A_j|)m < n(m - a_x)$. As $x \notin A_j$. Therefore:

$$1 = \sum_{x \in S} \frac{1}{n} = \sum_{x \in S} \sum_{A_j : x \notin A_j} \frac{1}{(n(m - r_x)} > \sum_{A_j} \sum_{x : x \not A_j} \frac{1}{(n - |A_j|)m} = \sum_{A_j} \frac{1}{m} = 1$$

which is a contradiction. QED.

There are many other proofs of Theorem 4, including one using linear algebra. However, this is by far the most beautiful one. We have discussed the Sylvester - Gallai theorem. We have looked at the related Slope Problem. We even generalized the Sylvester - Gallai theorem. However, we are not done. We must look at a related problem from graph theory. The proof is not as nice as the previous one but still quite elegant. It contrasts with the previous theorem as no combinatorial proof is known. Here it is:

Theorem 5: If a complete graph on n vertices is decomposed into complete bipartite subgraphs $H_1, H_2 ... H_n$, then $m \geq n - 1$.

Proof. To prove this theorem, we will set up a system of equations and use the solutions of the system of equations that lead to a contradiction. Let us call the complete graph K. Let us number the vertices 1 through n. Let T_j and B_j be the defining sets of vertices of a subgraph H_j. Assume toward a contradiction that $m < n - 1$. Take the equation:

$$\sum_{k=1}^{n} x_k = 0$$

and:

$$\sum_{k \in L_a} x_k = 0$$

for all a. By our assumption this system of equations must have integer solutions. Let us call the solution to this system of equations $s_1, s_2, ..., s_3$. Every vertex i has a corresponding variable v_i. As $H_1, H_2 ... H_3$ decompose K,

$$\sum_{i<j} x_i x_j = \sum_{k=1}^{m} (\sum_{t \in T_k} v_t \cdot \sum_{b \in B_k} v_b)$$

Therefore,

$$\sum_{i<j} c_i c_j = 0$$

Hence,

$$0 = (\sum_{k=1}^{n} c_k)^2 = \sum_{i=1}^{n} c_k^2 + 2\sum_{i<j} c_i c_j = \sum_{i=1}^{n} c_k^2 > 0$$

QED