Four/Five Colour Theorem

Harshikaa Agrawal

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Abstract

Starting from the basics of graph theory this paper explores the 5 colour theorem followed by the 4 colour theorem step by step as an attempt to enhance understanding of the two theorems.

1 Motivation

Suppose you are given a map to colour. What would be the minimum number of colours required to colour it? To visualise this problem more easily we can transform the map into a graph. If we denote each country as a vertex and each boundary as an edge we keep intact all necessary information. Here the problem would be to not have any two vertices that share an edge to have the same colour. This means we would have to chromatically colour the graph i.e. colour it so that we only use a minimum number of colours.

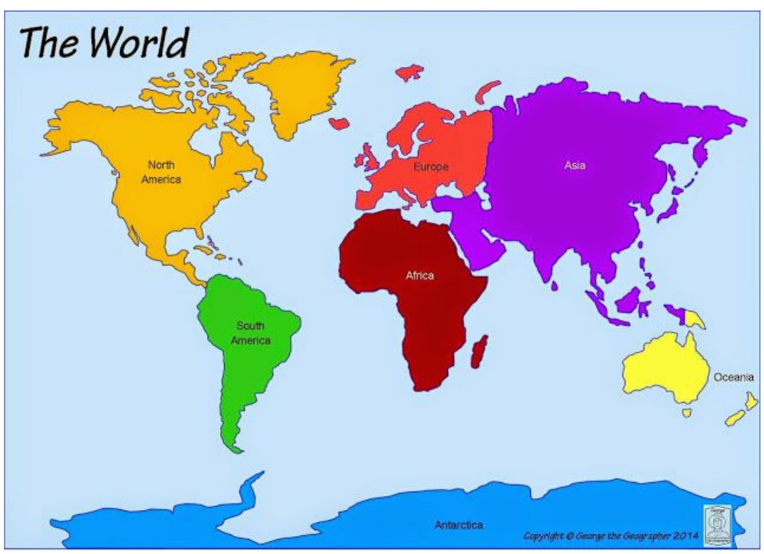

Figure 1: For the sake of simplicity consider this world map which has to be coloured continent wise.

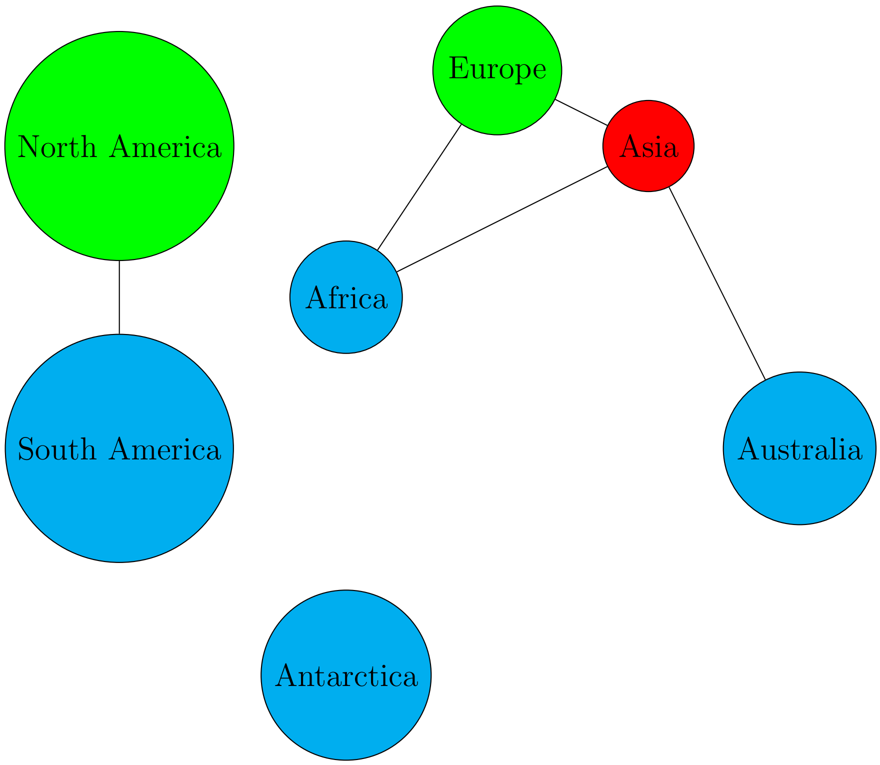

Figure 2: Map to Graph

2 Background

Graph Colouring is the colouring of vertices or edges in a graph such that no two adjacent nodes have the same colour. The Chromatic Number of a graph is the least number of colours needed to colour a graph. It is denoted as $\chi(G)$ or k-Chromatic-Graph, where k is the minimum number of colours required to colour the graph. In Figure 2, $\chi(G) = 3$ and it is a 3-Chromatic-Graph.

3 The Five Colour Theorem

Theorem 1. (Five Color Theorem) Every simple planar graph can be colored with 5 colors.

Lemma 2. Every Planar Graph contains a vertex with degree at most 5.

3.1 Understanding Thereom 1 and Lemma 2

A Graph is Planar when it can be drawn in the plane without any edges crossing it.

Theorem 3. When G is a connected planar graph which has with n vertices and m edges that divide it into r regions then $r = m - n + 2$

Proof. by Mathematical Induction.
Base Case: $m_1 = 1$ and $n_1 = 2$. $r_1 = m_1 - n_1 + 2 = 1$. Which is true since only 1 region is

there.

Induction Step: Assume the formula is true for some k such that $r_k = m_k - n_k + 2$ is true for Graph G_k.
Verification Step: Add an edge (u, v) to G_k results in a graph $G_k + 1$. Let $x = k + 1$

Case 1: Vertices u and v are already a part of G_k. Thus, $r_l = r_k + 1$, $n_l = n_k$ and $m_l = m_k + 1$.
$r_l = m_l - n_l + 2$
$r_k + 1 = m_k + 1 - n_k + 2$
$r_k = m_k - n_k + 2$
Thus the formula has been proven for case 1.

Case 2: v is a vertex added to G_k. Thus, $r_l = r_k$, $n_l = n_k + 1$ and $m_l = m_k + 1$.
$r_l = m_l - n_l + 2$
$r_k = m_k + 1 - n_k - 1 + 2$
$r_k = m_k - n_k + 2$
Thus the formula has been proven for case 2. □

Lemma 4. For a planar graph $3r \leq 2m$.

Proof. All planar graphs can be turned into maximal planar graphs or a simple planar graph by adding edges to the graph so that all regions are bound by 3 edges and the graph remains planar. Since the number of edges in a maximal graph are the maximum number of edges possible to have in a planar graph we are going to compare it to the number of edges in a complete graph. A complete graph has the maximum number of edges in a graph, all vertices have an edge between them. For ease of understanding we will double count both sides of the equation.
In a triangulation or maximal graph each side is bounded by three edges thus the number of edges are $3r$. The total number of edges in a complete graph would be equal to $n(n-1)$ which is equal to $2m$. Thus $3r \leq 2m$. This is true because a planar graph can never have more edges than a complete graph. □

Theorem 5. k_5 is not planar.

Proof. The theorem means that a complete graph of 5 vertices is not planar. k_5 is read as a complete graph of 5 vertices.
When $n = 5$, $2m = n(n-1) = 5(4) = 20$.
$r = m - n + 2 = 10 - 5 + 2 = 7$, $3r = 3(7) = 21$.
By lemma 4, $3r \leq 2m$ and $21 \leq 20$ is a contradiction. Thus, k_5 is non planar. □

Lemma 6. $m \leq 3n - 6$

Proof. $r = m - n + 2$, $3r = 3m - 3n + 6$
$3r \leq 2m$ (Lemma 4)
$3m - 3n + 6 \leq 2m$
$m \leq 3n - 6$ □

Theorem 7. If G is planar then G has a vertex of degree at most 5.

Proof. G is a connected component. If degree of all vertices is greater than 5 then $2m \geq 6n$. By lemma 6, $6n - 12 \geq 2m$.
$6n \leq 2m \leq 6n - 12$ is a contradiction. Thus there does exist a vertex of degree at most 5 in a planar graph.
(Theorem 7 is Lemma 2.) □

3.2 Proof by Mathematical Induction

Proof. Base Step: n $\leq$ 5, where n is the the number of vertices in planar graph G. This is true because the number of colours available aren't less than the number of vertices. Thus each vertex can have a unique colour.

Induction Step: vertex v has less than 5 vertices connected to it i.e. $deg(v) \leq 4$. Thus $k = 4$. G is 5-colourable since v plus the number of vertices it is connected to is 5 and we have 5 colours.

Verification for $k + 1$: Here $deg(v) = 5$. First assign a unique colour to the 5 vertices and leave v uncolored. Since we have already used all 5 colours for the graph we would have to repeat a colour for v. Assume that the 5 vertices are arranged in clockwise direction around v in the order v_1, v_2, v_3, v_4, v_5.
If v_1 and v_3 are disconnected then either of them can be coloured in a reverse way so that they both have the same colour and v is coloured by one of the colours that previously v_1 or v_3 was coloured with. .
If v_1 and v_3 are connected then v_2 and v_4 aren't connected, to preserve the planarity of G since it would cause the edges to cross. Thus there must exist some v_k and v_m that doesn't have an edge between themselves so that v can be coloured. Thus we have verified for $k+1$ which concludes the proof of the 5 colour theorem. □

4 The Four Colour Theorem

Theorem 8. The chromatic number of a planar graph is at most 4.

4.1 Comparing the 5 colour theorem to the 4 colour theorem

4.1.1 Why different proofs?

The v_5 in the proof of the 5 colour theorem has little significance so a natural question would be why can't the 4 colour theorem be proven using the same proof. Logically the same proof is valid however in the case of the four colour theorem lemma 2 isn't necessarily true for v. Thus the 4 colour theorem requires a different proof.

4.1.2 Why is the proof of the 4 colour theorem harder?

Due to the lack of lemma 2 the number of configurations (unique cases/possibilities) is more than 600 for the 4 colour theorem which is way more than 3 cases we used to prove the 5 colour theorem. The problem becomes much harder when the number of configurations to be considered increase to such high numbers.

4.2 What does the proof essentially do?

A set of graph configurations such that any smallest counterexample of the 4 colour theorem must contain at least a graph as a sub graph is known as an unavoidable set. And a reducible configuration is a graph with a reducible configuration which can be reduced to a smaller graph, satisfying the condition that if a sub graph can be colored with 4 colors, then the original graph can also be colored with 4 colors.
If an unavoidable set of reducible configurations can be found, then it would be enough to show that every graph in that set can be 5-colored. This means that if a counterexample must contain one of a set of graphs as a sub graph, but that if any of those sub graphs were part of a counterexample, a smaller counterexample existed. This demonstrated the result by showing that there cannot be any smallest counterexample, so there cannot be any counterexample at all.

In 1976 Appel Haken managed to prove the theorem for the first time in the world with 1936 configurations which were reduced by Robertson et al to approximately 600 configurations in 1997.

4.3 Use of Computers

The proof of the 4 colour theorem is extremely elaborate and relies on the computer. Since the proof requires an exhaustive search a computer decreases the risk of mistakes and makes the proof faster. Other than efficiency and accuracy there isn't any special reason the proof requires the use of computers.
This proof cannot be proven by hand because we do not know such a proof yet. The ones we do know are too long and tedious to do manually. It's possible that there is no short and concise proof of this theorem, regardless of which a computer assisted proof is in no way inadequate.

4.4 Applications

Though the motivation of the theorem is of map colouring, the application of it isn't actually used for colouring maps. It's actually used in mobile phone masts for signal and other such things.

5 Conclusion

Both theorems have some similar conditions and even though are loosely related vary greatly in terms of their proofs. While one can be proved quite easily the other requires a computer and hours of time.

References

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